(Weinberg 19.2)

• Goldstone Bosons •

Consider a Spontaneously Broken Continuous Symmetry. In field space, configurations obtained from the vacuum one by a global (constant) symmetry transformation have the same E as the vacuum:

$$\phi_{vac}(x) = \phi_{vac}(0) \longrightarrow e^{i\alpha_i(\vec{x}) Q^i}\, \phi_{vac} \equiv \phi'(x)$$

For $\alpha(\vec{x})$ slowly-varying $(\lambda \to \infty)$, $E \to E_{vac} = 0$

$\Downarrow$

"Fluctuations along Broken Generators are massless particles $\equiv$ 'Goldstone Bosons'"

Let us prove the $\exists$ of Goldstones:

I proof : Be $V[\phi]$ the effective potential

ϕ_n = hermitian scalar field

$$\underbrace{\phi_n(x) \longrightarrow \phi_n(x) + i\,\varepsilon^A t^A_{nm}\, \phi_m(x)}_{\text{Under Symmetry Group } G}$$

• Effective Pot. is invariant:

$$\sum_{n,m} \frac{\partial V(\phi)}{\partial \phi_n} t^A_{nm} \phi_m = 0$$

take another $\partial/\partial\phi_e$, and fix $\phi = \phi_{vac}$

$$\sum_{n,m} \frac{\partial^2 V(\phi_{vac})}{\partial\phi_e \partial\phi_n} t^A_{nm} \phi^{vac}_m + \cancel{\sum_n \underbrace{\frac{\partial V(\phi_{vac})}{\partial \phi_n}}_{\overset{\parallel}{0}} t^A_{ne}} = 0$$

on vacuum

$$\mu^2_{en} = \frac{\partial^2 V}{\partial\phi_e \partial\phi_n}(\phi_{vac}) = \text{"mass-matrix"}$$

$(v^A)_n \equiv t^A_{nm} \phi^{vac}_m$ has zero eigenvalue

1 zero eigenvalue $\forall \hat{A} \,/\, t^{\hat{A}} \cdot \phi^{vac} \neq 0$

$\{t^{\hat{A}}\} =$ "broken generators"

Exercise (1)

$$\mathcal{L} = \frac{1}{2} \sum_{n=1,2} \partial_\mu \phi_n \partial^\mu \phi_n + \frac{\mu^2}{2} \sum_n \phi_n^2 - \frac{g}{4}\left(\sum_n \phi_n^2\right)^2$$

work out the spectrum for $\mu^2 > 0$, how many Goldstones do I get? What is the symmetry breaking pattern? What would the spectrum be for
$n = 1, 2, \ldots, N$?

Exercise 2:

Given an arbitrary symmetry group G, the generators that are unbroken always form a subgroup H of G. Can you show that?

The existence of Goldstones: II proof

Consider one of the broken generators and its associated current:

$$J^{\mu \hat{A}} \; : \; \partial_\mu J^{\mu \hat{A}} = 0$$

$$Q^{\hat{A}} \equiv \int d^3x \, J^{0 \hat{A}}(\vec{x}, 0) \quad \text{is such that:}$$

$$Q^{\hat{A}} |0\rangle \neq 0 \, , \quad \text{and}$$

$$\left[Q^{\hat{A}}, \phi_n(x) \right] = - \sum_m t^{\hat{A}}_{nm} \phi_m(x)$$

where

$$t^{\hat{A}}_{nm} \langle 0 | \phi_m | 0 \rangle \neq 0$$

You can think to ϕ_n as any of the local operators in the theory whose VEV break the symmetry. ϕ_n might not be a fundamental field.

In QCD: $\phi \cong \bar{q}_L q_R + h.c.$

Suppress $\hat{A}$ and m index and compute

$$\langle 0|\left[J^\mu(y),\, \phi(x)\right]|0\rangle =$$

$$= (2\pi)^{-3}\int d^4P\left[i\rho^\mu(P)\, e^{iP(y-x)} - i\tilde{\rho}^\mu(P)\, e^{iP(x-y)}\right]$$

where we have rewritten

$$\langle 0|J^\mu(y)\,\phi(x)|0\rangle = \sum_N \langle 0|J^\mu(y)|N\rangle\langle N|\phi(x)|0\rangle$$

$$= \sum_N \langle 0|J^\mu(0)|N\rangle\langle N|\phi(0)|0\rangle\, e^{iP_N y}\, e^{-iP_N x}$$

$\hookrightarrow$ this denotes the completeness sum of the complete set $|N\rangle$

$$(2\pi)^{-3} i\rho^\mu(P) \equiv \sum_N \langle 0|J^\mu(0)|N\rangle\langle N|\phi(0)|0\rangle\, \delta^4(P-P_N)$$

$$(2\pi)^{-3} i\tilde{\rho}^\mu(P) \equiv \sum_N \langle 0|\phi(0)|N\rangle\langle N|J^\mu(0)|0\rangle\, \delta^4(P-P_N)$$

where P_N (to be $\int$ over by the $\sum_N$) runs over physical multi-particle states.

By Lorentz Invariance: (P = physical as P_N is, has $P_0>0,\, P^2>0$)

$$\rho^\mu = P^\mu \rho(P^2)\,\sigma(P_0)$$
$$\tilde{\rho}^\mu = P^\mu \tilde{\rho}(P^2)\,\sigma(P_0)$$

Therefore

$$\langle 0 | [J^\mu(y), \phi(x)] | 0 \rangle =$$

$$= \frac{\partial}{\partial y_\mu} \left[(2\pi)^{-3} \int d^4 p \left[\rho(p^2) e^{ip(y-x)} + \tilde{\rho}(p^2) e^{ip(x-y)} \right] \cdot \sigma(p_0) \right]$$

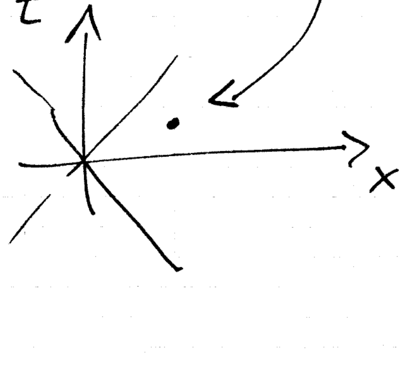
$$= \frac{\partial}{\partial y_\mu} \left[\int d\mu^2 \left[\rho(\mu^2) \Delta_+(y-x;\mu^2) + \tilde{\rho}(\mu^2) \Delta_+(x-y;\mu^2) \right] \right]$$

$$\Delta_+(z;\mu^2) = (2\pi)^{-3} \int d^4 p \, \sigma(p^0) \, \delta(p^2-\mu^2) e^{ipz}$$

If $y-x$ is space-like, causality makes the $[\,,\,]$ vanish. Moreover

$$\Delta_+(z,\mu^2) = \Delta_+(-z,\mu^2)$$

for z space-like (sign(z_0) not invariant)



$\Downarrow$

$$\Delta_+ = f(z^2,\mu^2)$$

this $\Rightarrow \quad \rho = -\tilde{\rho}$

$$\Rightarrow \quad \langle 0 | [J^\mu(y), \phi(x)] | 0 \rangle =$$

$$= \frac{\partial}{\partial y_\mu} \int d\mu^2 \rho(\mu^2) \left[\Delta_+(y-x;\mu^2) - \Delta_+(x-y;\mu^2) \right]$$

this is of course $\neq 0$ at time-like distance

Remember now:

$$\partial_\mu J^\mu = 0 \quad \text{and} \quad \Box \Delta_+ = -\mu^2 \Delta_+$$

$$\Rightarrow \int d\mu^2 \rho(\mu^2) \, \mu^2 \left[\Delta_+(y-x;\mu^2) - \Delta_+(x-y;\mu^2) \right] \overset{!}{=} 0$$

$$\Rightarrow \quad \boxed{\mu^2 \rho(\mu^2) = 0}$$

However, ρ itself cannot be zero:

$$\langle 0 | [J^0(\vec{y},t), \phi(\vec{x},t)] | 0 \rangle =$$

$$= (2\pi)^{-3} \int d^4p \, i P^0 \rho(P^2) \sigma(P_0) \left[e^{i\vec{P}(\vec{y}-\vec{x})} + e^{i\vec{P}(\vec{x}-\vec{y})} \right]$$

$$= (2\pi)^{-3} \int d\mu^2 \rho(\mu^2) \int d^4p \, i P^0 \sigma(P_0) \delta(P^2 - m) 2 e^{i\vec{P}(\vec{x}-y)}$$

But we have:

$$\text{"}\delta(P_0^2 - \vec{P}^2 - m^2)\sigma(P_0) = \delta(P_0 - \sqrt{\vec{P}^2 + m^2}) \frac{1}{2P_0}\text{"}$$

$$= (2\pi)^{-3} \int d\mu^2 \rho(\mu^2) \cdot i (2\pi)^3 \delta^3(\vec{x}-\vec{y})$$

$$\Rightarrow \langle 0 | [Q^{\hat{A}}, \phi_n] | 0 \rangle = i \int d\mu^2 \rho_n^{\hat{A}}(\mu^2) = -\sum_m t^{\hat{A}}_{nm} \langle \phi_m \rangle$$

We eventually find:

$$\rho_n^{\hat{A}}(\mu^2) = i \delta(\mu^2) \sum_m t^{\hat{A}}_{nm} \langle 0 | \phi_m(0) | 0 \rangle$$

where, remember

$$(2\pi)^{-3} i \rho^\mu(P) = \sum_N \langle 0 | J^\mu(0) | N \rangle \langle N | \phi(0) | 0 \rangle \delta(P - P_N)$$

The structure $\rho \sim \delta$ can only be due to a massless 1-particle state contribution.

Let us settle our conventions ($\neq$ from Weinberg's)

$$[a(\vec{p}), a^+(\vec{q})] = (2\pi)^3 2E_p \delta^3(\vec{p}-\vec{q}) \qquad \ulcorner \Rightarrow [a] = -1 \lrcorner$$

$$\langle \psi_{\vec{p}} | \psi_{\vec{q}} \rangle = (2\pi)^3 2 E_p \delta^{(3)}(\vec{p}-\vec{q})$$

$$\Downarrow$$

$$\mathbb{1} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\psi_{\vec{p}}\rangle\langle\psi_{\vec{p}}| \quad ; \quad \boxed{\text{For massless particle: } E_p = |\vec{p}|}$$

The 1-particle state contribution to ρ^μ then reads $\left(\sum_N \to \int \frac{d^3q}{(2\pi)^3 2E_q} \right)$

$$(2\pi)^{-3} i\rho^\mu(p) = (2\pi)^{-3} \int \frac{d^3q}{(2E_q)} \langle 0 | J^\mu(0) | B\vec{q}\rangle \langle B\vec{q} | \phi(0) | 0 \rangle \cdot \delta(p_0 - E_q)\, \delta^3(\vec{p}-\vec{q})$$

$$i\rho^\mu(p) = \frac{1}{2|\vec{p}|} \delta(p_0 - |\vec{p}|) \langle 0 | J^\mu | B, \vec{p}\rangle \langle B, \vec{p} | \phi(0) | 0 \rangle$$

$$= \theta(p_0)\, \delta(p_0^2 - |\vec{p}|^2) \langle 0 | J^\mu | B, \vec{p}\rangle \langle B, \vec{p} | \phi(0) | 0 \rangle$$

We parametrize the two matrix elements as:

$$\langle 0 | J^\mu(x) | B, \vec{P} \rangle = i F p^\mu e^{iPx} \qquad [\,[F] = 1\,]$$

$$\langle B | \phi_n | 0 \rangle = Z_n e^{iPx}$$

$$\Rightarrow \quad i\rho^\mu(P) = \sigma(P_0)\, \delta(P^2)\, i P^\mu F Z_n$$

$$\Rightarrow \quad F Z_n = i \sum_m t_{nm} \langle \phi_m \rangle \neq 0$$

We then conclude

① $|B\rangle$ is a $J=0$ Boson, because

$$\langle B | \phi_n | 0 \rangle \neq 0$$

② $|B\rangle$ has the same unbroken QN as the broken generators

③ In general, there is one Goldstone for each broken generator

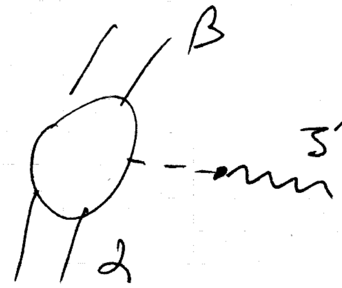
$$\langle 0 | J^{\mu,\hat{A}}(0) | B^{\hat{B}} \rangle = i F^{\hat{A}\hat{B}} p^\mu$$

We will see that symmetries constrain GB interaction a lot. For the moment, let us just notice this:

$$\langle \beta | J^{\mu}(x) | \alpha \rangle = \langle \beta | J^{\mu}(0) | \alpha \rangle \, e^{iqx}$$

$$q = P_{\alpha} - P_{\beta}$$

this has only one pole at $q^2 = 0$, coming from:



$$\langle \beta | J^{\mu}(0) | \alpha \rangle = N^{\mu} + \frac{iF}{q^2} \cdot q^{\mu} \cdot M$$

$$0 = q_{\mu} J^{\mu} \implies M = i \, \frac{q_{\mu}}{F} N^{\mu}$$

$$\Downarrow$$

Goldstones have derivative interactions:

"Goldstones Coupling $= E/F$"