(Weinberg 19.4)

# • Pions As Goldstone Boson •

The Goldstone Theorem leads to its more brilliant physical consequences when applied to the QCD theory of Strong Interactions. Not only it explains the existence of the lightest hadrons, the pions, it also allows to control their dynamics, extending, in a sense, our predictive power to the far IR where the QCD theory becomes non-perturbative.

To a good approximation, the $u$ and $d$ quarks are massless, and the QCD Lagrangian is:

will not be relevant

$$\mathcal{L}_{QCD} = \bar{u}\, i \not{D} u + \bar{d}\, i \not{D} d \left( -\frac{1}{2} \mathrm{tr}\left[G^2\right] \right)$$

where $(D_\mu)^{ij} = \delta^{ij}\partial_\mu - i g_s A^A_\mu \left(\frac{\lambda^A}{2}\right)^{ij}$ ; $i, j$ are $SU(3)_c$ indeces

You surely know about the $SU(3)_c$ color invariance of $\mathcal{L}_{QCD}$, today we discuss a different symmetry group, a flavor symmetry $SU(2)_L \times SU(2)_R$

Split the Dirac spinors in chiral components:

$$u = \frac{1}{2}(1+\gamma_5)u + \frac{1}{2}(1-\gamma_5)u = u_L + u_R$$

$$d = \cdots = d_L + d_R$$

$$\mathcal{L}_{QCD} = \bar{u}_L i \not{D} u_L + \bar{d}_L i\not{D} d_L + \bar{u}_R i \not{D} u_R + \bar{d}_R i \not{D} d_R$$

Define $\quad q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \qquad q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix}$

And perform independent SU(2) rotations on $q_L$ and $q_R$:

$$\begin{bmatrix} q_L \\ q_R \end{bmatrix} \longrightarrow \begin{bmatrix} e^{i\vec{\sigma}_L \cdot \vec{\tau}} & 0 \\ 0 & e^{i\vec{\sigma}_R \vec{\tau}} \end{bmatrix} \begin{bmatrix} q_L \\ q_R \end{bmatrix}$$

where $\vec{\tau}$ are Isospin matrices

$$\tau_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \tau_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\tau_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Lagrangian is invariant

## Question:

If I define $q_L^c = (q_R)^c$ by charge conjugation I get a Weyl Left-handed fermion, and I could consider more general transformations that mix $q_L$ and $q_L^c$.

Why those are forbidden in QCD? What if $SU(3)_c$ was instead $SO(3)_c$? Which global group would I have?

In this 4-plet notation the generators are

$$\vec{t}_L = \begin{bmatrix} \vec{\tau} & 0 \\ 0 & 0 \end{bmatrix}; \quad \vec{t}_R = \begin{bmatrix} 0 & 0 \\ 0 & \vec{\tau} \end{bmatrix}$$

the algebra is:

$$[t_L^i, t_L^j] = i\,\varepsilon^{ijk} t_L^k$$
$$[t_R^i, t_R^j] = i\,\varepsilon^{ijk} t_R^k$$
$$[t_L^i, t_R^j] = 0$$

$SU(2)_L \times SU(2)_R$ Global Group

NOTE: Additional $U(1)_L \times U(1)_R = U(1)_B \times U(1)_A$,
B = "baryon Number", we will come back ~~to~~ this

The $SU(2)_V$ subgroup consits of equal L and R $SU(2)$ rotations. Its generators are

$$\vec{t} \equiv \vec{t}_L + \vec{t}_R$$

Let us also define "axial" generators

$$\vec{x} = \vec{t}_L - \vec{t}_R$$

The new commutation relations are:

$$[t_i, t_j] = i\,\varepsilon_{ijk}\, t_k$$
$$[t_i, x_j] = i\,\varepsilon_{ijk}\, x_k$$
$$[x_i, x_j] = i\,\varepsilon_{ijk}\, t_k$$

By Noether's Theorem, all these symmetries lead to 6 independent conserved currents and charges:

$$\vec{J}_L^{\mu} = \bar{q}_L \gamma^\mu \vec{t}\, q_L \;;\quad \vec{J}_R^{\mu} = \bar{q}_R \gamma^\mu \vec{t}\, q_R$$

or, in the $\vec{t}, \vec{x}$ basis:

$$\vec{V}^{\mu} = \vec{J}_L^{\mu} + \vec{J}_R^{\mu} = \bar{q}\gamma^\mu \vec{t}\, q$$
$$\vec{A}^{\mu} = \vec{J}_L^{\mu} - \vec{J}_R^{\mu} = \bar{q}\gamma^\mu \gamma_5 \vec{t}\, q$$

Conserved Isospin and X charges are

$$\vec{T} = \int d^3x \, \vec{V}^0 \quad ; \quad \vec{X} = \int d^3x \, \vec{A}^0$$

$$[T_i, T_j] = i\,\varepsilon_{ijk} T_k$$

$$[T_i, X_j] = i\,\varepsilon_{ijk} X_k$$

$$[X_i, X_j] = i\,\varepsilon_{ijk} T_k$$

Observed hadrons form almost degenerate multiplets of $\vec{T}$ (isospin), so $\vec{T}$ is really a symmetry.

If $\vec{X}$ was also a symmety, any $|h>$ would be degenerate with states like $\vec{X} \cdot |h>$, notice that $\vec{X} \cdot |h> \neq 0$ if $\vec{T} \cdot |h> \neq 0$. But $\vec{X}$ is a pseudo-scalar operator:

$$P \vec{X} P^{-1} = -\vec{X}$$

We would observe degenerate multiplets made of particles of opposite parity, and we do not.

To avoid conflict with observations, the $\vec{X}$ generators must be spontaneously broken in QCD. To present, this "Spontaneous Chiral Symmetry Breaking" ($\chi_{SB}$) has not been proven, but is confirmed by many observational evidences.

Assuming $\chi_{SB}$ we have that 3 Goldstone Bosons exist, with the Quantum Numbers of the broken charge $\vec{X}$:

$$\vec{\pi} = \{ \pi^1, \pi^2, \pi^3 \} \quad \text{Are an } SU(2)_V \text{ triplet with } P(\pi) = -1$$

These particles exist, are the Pions. However they are not massless: $m_\pi$ is "small" but $\neq 0$

$$m_\pi \simeq 140 \text{ MeV}$$

The 3 pions are created from the vacuum by the axial current:

$$\langle 0 | A^\mu_i(x) | \pi_j \rangle = \frac{F_\pi}{2} \delta_{ij} \, p^\mu_\pi \, e^{i p_\pi \cdot x}$$

As again implied by the Goldstone Theorem

The Goldstone theorem does not tell us what $F_\pi$ is. We might in principle compute it from QCD, but only with complicated Lattice Simulation methods. It is better to measure it, from the Pion's decay width:

$$\Gamma(\pi^+ \to \mu^+ + \nu_\mu) = (2.6 \cdot 10^{-8} s)^{-1}$$

The Decay is indeed mediated by the Fermi Lagrangian:

$$\mathcal{L}_{Fermi} = \frac{G_{wk}}{\sqrt{2}} \left( V_+^\mu + A_+^\mu \right) \sum_{e=e,\mu} \bar{e}\, \gamma_\mu (1+\gamma_5) \nu_e + h.c.$$

$$V_\pm = V_1 \pm i V_2 \qquad A_\pm = A_1 \pm i A_2$$

$G_w$ is measured $G_{wk} = 1.15 \cdot 10^{-5}\, GeV^{-2}$

$$\mathcal{M}[\pi^+ \to \mu^+ + \nu_\mu] \propto \langle 0|_h \langle \mu^+ \nu_\mu|_e \frac{G_{wk}}{\sqrt{2}} (V_+^\mu + A_\mu^\mu) \cdot$$
$$\times\, \bar{\mu} \gamma_\mu (1+\gamma_5) \nu_\mu |0\rangle_e |\pi^+\rangle$$
$$= \frac{G_{wk}}{\sqrt{2}} \langle 0|_h (V + A) |\pi^+\rangle \times$$
$$\langle \mu^+ \nu_\mu|_e \bar{\mu} \gamma_\mu (1+\gamma_5) \nu_\nu |0\rangle_e$$

One finds: $\Gamma \propto F_\pi^2$, from which

$$F_\pi \simeq 184 \text{ MeV}$$

**Exercise**: Show that the pion width is

$$\Gamma(\pi \to \mu \nu) = \frac{G_{wk}^2 \, F_\pi^2 \, m_\mu^2 \left(m_\pi^2 - m_\mu^2\right)^2}{16 \pi \, m_\pi^3}$$

and use the above formula to extract the value of $F_\pi$

Having measured $F_\pi$, we would like to use its value to make some prediction. Let us then consider another process, the neutron decay. The relevant matrix element is:

$$\langle p | A_\mu^+(0) | n \rangle = \bar{u}_p \left[ f(q^2) \gamma^\mu \gamma_5 + g(q^2) q^\mu \gamma_5 \right] u_n \qquad (1)$$

where $q = P_n - P_p$ and $u_{n,p}$ are the neutron's and proton's spinor wave-functions

The parametrization above is the most general one compatible with Lorentz invariance, P and C conservation.

Exercise : Show Equation (1)

Remember : Neglecting $m_\pi$, the chiral symmetry is an "exact" symmetry, in the sense that it is a symmetry of the Lagrangian. This makes that the axial current IS conserved :

$$q^\mu A_\mu^+ = 0 \Rightarrow \bar{u}_p \left[ f(q^2)\, \underset{0}{\underset{\shortparallel}{\not{q}\, \gamma_5 + g(q^2) q^2 \gamma_5}} \right] u_n$$

The EOM for the spinors, neglecting p-n mass-difference, is

$$\not{p}_p u_p + m_N u_p = 0$$

$$\not{p}_n u_n + m_N u_n = 0$$

$$\Rightarrow \bar{u}_p \not{q} \gamma_5 u_n = \bar{u}_p \not{p}_n \gamma_5 u_n + \bar{u}_p \not{p}_p \gamma_5 u_n$$

$$= -2 m_N \bar{u}_p \gamma_5 u_n$$

$$\Rightarrow 2 m_N f(q^2) = q^2 g(q^2)$$

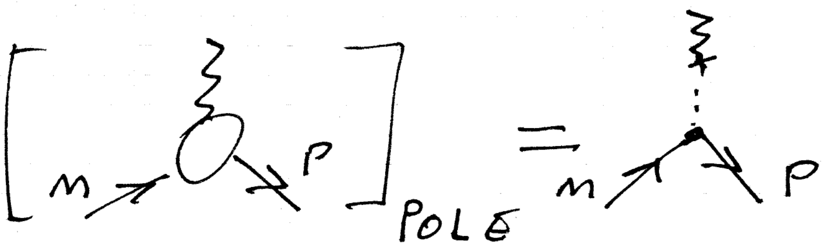
$$\Rightarrow g(q^2) \xrightarrow{q^2 \to 0} \frac{2 m_N f(0)}{q^2}$$

$g(q^2)$ must have a pole at $q^2 \to 0$.

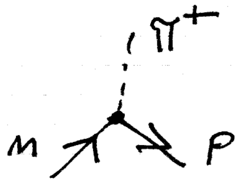
At $q^2 \to 0$,

$$f(0) = g_A = 1.26$$

Measured in $n \to \pi e \nu$ decay, similarly to $F_\pi$. We then know that $g$ has a massless pole, this is due to the exchange of a pion:



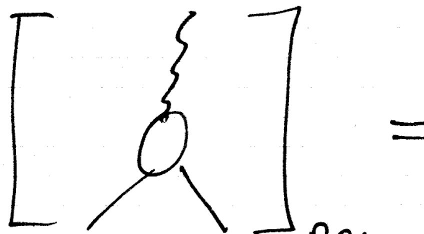

the vertex




$$\equiv 2 G_{\pi N} \gamma_5$$

comes from the Lagrangian $2i G_{\pi N} \vec{\pi} \bar{N} \gamma_5 \vec{\tau} N$ and can be measured independently, in $\pi N \to \pi N$ scattering processes.

$$\frac{2 m_N g_A}{q^2} \underbrace{\bar{u}_p \hat{q} \gamma_5 u_n} =$$




$$= \frac{2 F_\pi G_{\pi N}}{2} \frac{\bar{u}_p \hat{q} \gamma_5 u_n}{q^2}$$

$$\Rightarrow G_{\pi N} = \frac{2 m_N g_A}{F_\pi}$$

"Goldberger-Treiman Relation"

Experimentally, $G_{\pi N} \cong 13.4$, the G-T relation holds to $\sim 10\%$ level. This provided, historically, an important confirmation to the picture of Goldstone Bosons.

Proceeding with similar methods, many more relations might be derived, but we see this is rather complicated. Also, the effects of explicit $\chi_{SB}$ (i.e. mass of the quarks, which will lead to a mass for the pions) might be incorporated (read Weinberg if interested). But both tasks are much more easily achieved by an "Effective Field Theory" approach, which will lead us to a phenomenological Lagrangians for the pions.