WEINBERG 13.5
GEORGI 2.6

• An Effective Theory of pions • (I)

The Goldstone Theorem, plus standard symmetry considerations, has allowed us to predict the $G_{\pi NN}$ coupling, which controls the low-energy pion-Nucleon scattering Amplitudes. It also allowed us to have some control (see also Weinberg 19.2) on the amplitude for a single Goldstone emission. That result was based on the current conservation and on the fact that the Goldstone state is created from the vacuum by this conserved current. Similar considerations might be applied (and were indeed applied in the past) to compute amplitudes with more Goldstones. This technique was "standardized" and took the name of "current Algebra" technique.

Whatever this current algebra was, it was only based on symmetries and on the Goldstone Theorem. All its results are therefore totally insensitive to the detailed structure of the theory (QCD) which underlies the pion's interactions. They only depend on the symmetry and on the symmetry breaking pattern, they should hold exactly in the same way in any other theory with the same features.

These considerations led to the formulation of an effective Theory for the pions. The idea was to build the simplest possible model with the same symmetry breaking pattern as QCD. In this model the current algebra predictions are authomatically incorporated in the leading order (tree-level) calculations. It is however much simpler to obtain these predictions from the effective field theory calculations, rather than the messy current algebra manipulations. It was later realized that the effective field theory approach is predictive beyond Current Algebra. Even Next-to-Leading order (loop) corrections can be computed in the effective theory.
Realizing this required a major change in the approach to QFT.

The symmetry breaking pattern is, in QCD:

$$SU(2)_L \times SU(2)_R \longrightarrow SU(2)_V$$

This breaking comes from the VEV of a scalar operator X, transforming in the $(2,2)$:

$$X \longrightarrow g_L X g_R^+ \quad ; \quad g_{L,R} \in SU(2)_{L,R}$$

This operator takes a vev

$$X \propto \mathbb{1} \Rightarrow SU(2)_V \; (g_L = g_R)$$

subgroup survives, all the rest is broken

What is X in QCD? It is important to know that <u>we</u> <u>do</u> <u>not</u> <u>really</u> <u>know</u>! But we have many many candidates:

$$X_{ij} \overset{?}{=} \bar{q}_{R,j} q_{L,i} \overset{?}{=} \Box \bar{q}_{Rj} q_{Li} \overset{?}{=} (G_{\mu\nu})^2 \bar{q} q = \dots \text{ etc}$$

Moreover, local operators need to be <u>defined</u> (renormalized) in QFT, all what matters is the quantum numbers of X, and in particular the fact that it is in the $(2,2)$

• For what concerns the Goldstone theorem, a model with any matrix of scalar fields in the $(2,2)$, with VEV $\langle \phi \rangle \propto \mathbb{1}$, would behave the same as QCD. ϕ does not need to be a generic $(2,2)$ matrix, we can impose any restriction that does not break the $SU(2)\times SU(2)$ symmetry, let us take:

$$\phi = \phi^c \equiv \sigma_2 \phi^* \sigma_2$$

This condition is $SU(2)_L \times SU(2)_R$ - invariant:

$$\sigma_2 \phi^* \sigma_2 \longrightarrow \sigma_2 g_L^* \sigma_2 \sigma_2 \phi^* \sigma_2 \sigma_2 g_R^t \sigma_2 =$$

$$= g_L \sigma_2 \phi^* \sigma_2 \, g_R^+$$

because, remember

$$\sigma_2 \sigma_i^* \sigma_2 = -\sigma_i$$

From here we see explicitely that

$$\phi \neq X \; !$$

If, for instance, $X = \bar{q} q$, $X^c \neq X$, the field ϕ is only required to have the right quantum numbers, has nothing to do with X!

As any 2x2 matrix, ϕ can be expanded as:

$$\phi = h_4 \cdot \mathbb{1} + i\, h_i \sigma^i$$

where $h_m = \{h_1, \dots, h_4\}$ are Real because of the condition $\phi = \phi^c$. You will see in an exercise that

$$SU(2)_L \times SU(2)_R \simeq SO(4)$$

The $g_{L,R}$ rotations of ϕ are equivalent to rotations $SO(4)$ on the 4-plet h_m.
The invariant Lagrangian is:

$$\frac{1}{4} \mathrm{tr}\left[(\partial_\mu \phi)^+ \partial^\mu \phi\right] + \frac{\mu^2}{4} \mathrm{tr}\left[\phi^+ \phi\right] + \\ - \frac{\lambda}{4}\left[\frac{1}{2} \mathrm{tr}\left[\phi^+ \phi\right]\right]^2 =$$

$$= \frac{1}{2} \sum_m \partial_\mu h_m \partial^\mu h_m + \frac{\mu^2}{2} \sum_m h_m^2 - \frac{\lambda}{4} \left(\sum_m h_m^2\right)^2$$

this is the theory we already discussed in an exercise, we know that

$$\langle h_m \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mu/\sqrt{\lambda} \end{pmatrix}; \quad \mu/\sqrt{\lambda} \equiv v$$

and that there are 3 massless Goldstone, like in QCD, and a massive field σ, unlike in QCD. The two theories resemble each other, as expected, at $E \to 0$ where only the Goldstones matter.

We could study the theory around the vacuum by employing a "linear" field basis:

$$h = \begin{pmatrix} q_\ell \\ v + \rho \end{pmatrix} \begin{matrix} \to \text{massless Goldstone} \\ \to \text{massive } \rho \text{ field} \end{matrix}$$

or a "non linear" one:

$$h_n \equiv R_{n4}(x)\,\sigma(x) \longrightarrow \left\{ \begin{array}{l} \text{Act on a "reference"} \\ \text{4-vector } \left(\begin{smallmatrix}0\\0\\0\\\sigma\end{smallmatrix}\right) \text{ with} \\ \text{a generic } SO(4) \\ \text{rotation } R \end{array} \right.$$

$$R^T R = \mathbb{1}$$

$$\mathcal{L} = \frac{1}{2}\left[R_{n4}\partial_\mu\sigma + \sigma\partial_\mu R_{n4}\right]^2 + \frac{\mu^2}{2}\sigma^2 - \frac{1}{4}\sigma^4$$

but $R_{n4}\partial_\mu\sigma\, R_{n4}\partial_\mu\sigma = (\partial\sigma)^2$ as $R^t R = \mathbb{1}$

and $\sigma\partial_\mu\sigma\, R_{n4}\partial_\mu R_{n4} = \sigma\partial_\mu\sigma \cdot \frac{1}{2}\partial_\mu\left[\sum_n (R_{n4})^2\right] = 0$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\sigma^2\partial_\mu R_{n4}\partial^\mu R_{n4} + \frac{\mu^2}{2}\sigma^2 - \frac{1}{4}\sigma^4$$

It is convenient to parametrize R_{m4}, which is such that $\sum_m (R_{m4})^2 = 1$, in this way:

$$R_{a4} = \frac{2\xi_a}{1+\vec{\xi}^2} \qquad R_{44} = \frac{1-\vec{\xi}^2}{1+\vec{\xi}^2}$$

$\xi_a = \xi_{1,2,3}$ will describe the 3 massless Goldstones

Substituting (check by yourselves as an exercise) the Lagrangian becomes:

$$\mathcal{L} = +\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + 2\sigma^2\vec{D}_\mu\cdot\vec{D}^\mu + \frac{\mu^2}{2}\sigma^2 - \frac{\lambda}{4}\sigma^4$$

$$\vec{D}_\mu \equiv \frac{\partial_\mu\vec{\xi}}{1+\vec{\xi}^2}$$

Remember that σ will take a VEV $v = \mu/\sqrt{\lambda}$; then we redefine:

$$\sigma \to \sigma + v$$

then given that

$$V[\sigma] = \frac{\lambda}{4}\left[\sigma^2 - v^2\right]^2 + \text{cst}$$

$$= \frac{\lambda}{4}\left[\sigma^2 + 2v\sigma\right]^2 = \lambda v^2\sigma^2 + \lambda v\sigma^3 + \frac{\lambda}{4}\sigma^4$$

$$\mathcal{L} = 2v^2 \vec{D}_\mu \cdot \vec{D}^\mu + 4v\sigma \vec{D}_\mu \cdot \vec{D}^\mu + 2\sigma^2 \vec{D}_\mu \cdot \vec{D}^\mu +$$
$$+ \lambda v^2 \sigma^2 + \lambda v \sigma^3 + \frac{\lambda}{4}\sigma^4 + \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma$$

From the above equation we can read first of all the masses of the particles:

$$m_g = 0 \qquad m_\sigma = \sqrt{2\lambda}\, v = \sqrt{2}\,\mu$$

We would have found of course the same masses if working in the linear basis, this is what you probably did when solving the Exercise 1.1. If you did, and you also computed the interaction term, you should have found

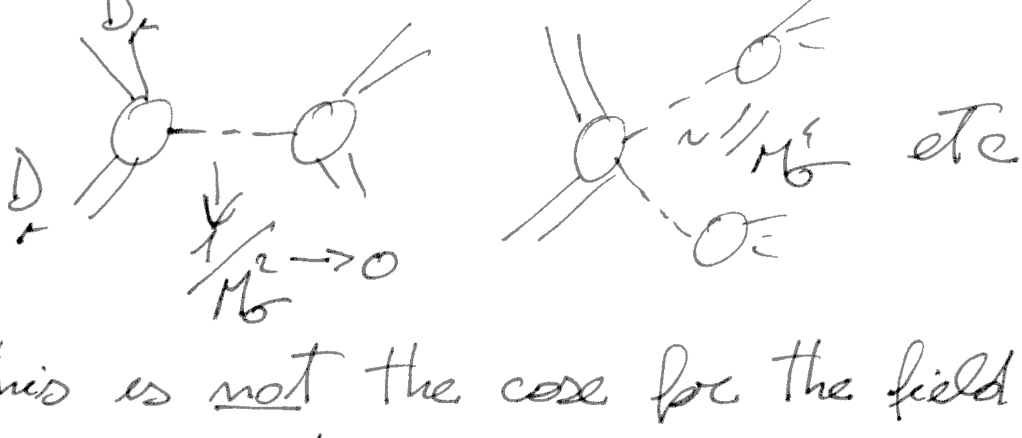
$$\tilde{\mathcal{L}} = \frac{1}{2}\partial_\mu \rho \partial^\mu \rho + \frac{1}{2}\partial_\mu \vec{\varphi}\, \partial^\mu \vec{\varphi} - \lambda v^2 \rho^2 +$$
$$+ \frac{\lambda}{4}\left[4v\rho^3 + 4v\rho\, \vec{\varphi}\cdot\vec{\varphi} + 2\rho^2\, \vec{\varphi}\cdot\vec{\varphi}\right]$$

$\tilde{\mathcal{L}}$ looks very different from $\mathcal{L}$. However it describes the same physics: $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are related by a field redefinition. The two Lagrangians describe the same on-shell amplitudes, even though they look so different!

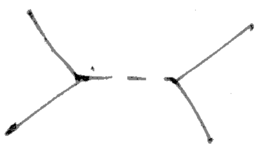
For our purposes, $\mathcal{L}$ is more useful. What we are interested in is, indeed, a model for the massless pions, not for σ. The particle σ (which we would call ρ in the other basis) is indeed a massive one. At the extremely low energy we are interested in $(E/F_\pi \to 0)$ The presence of σ as a physical propagating particle does not show up. We would like, then, to "ignore" σ, because it will be much heavier than the energy we are working at. Taking the limit

$$m_\sigma \to \infty \qquad \text{i.e. } \lambda \to \infty; \quad v = \text{finite}$$

we see that all diagrams involving σ can be ignored, in the non-linear basis:



while this is <u>not</u> the case for the field ρ in the linear basis:

 $$\sim \frac{\lambda^2 v^2}{m_\sigma^2} \sim \lambda \to \infty \,!$$

This means that in the linear basis one cannot simply ignore the field ρ, but instead follow a more complicated procedure called "integrating out" the ρ field. In the non-linear basis, instead, all is simpler.

A related fact is that the σ field is SO(4)-invariant:

$$\sigma = \sqrt{\sum_n h_n^2} - v$$

while ρ is not invariant. Notice that in spite of being the symmetry spont. broken, both the $\mathcal{L}$ and $\tilde{\mathcal{L}}$ Lagrangians are SO(4) invariant. If σ is itself invariant, the term without σ in $\mathcal{L}$ is also invariant by itself. Therefore, the restricted Lagrangian without σ field is:

$$\mathcal{L} = 2v^2 \vec{D}_\mu \vec{D}^\mu = 2v^2 \frac{\partial_\mu \vec{\varphi} \cdot \partial^\mu \vec{\varphi}}{(1+\vec{\varphi}^2)^2}$$

$$= \frac{1}{2} \frac{\partial \vec{\pi} \cdot \partial \vec{\pi}}{(1+\vec{\pi}^2/F^2)^2}$$

having defined $\varphi = \pi/2v$; $2v = F$

In the end, after putting σ to zero, we just got what is called a "Non-Linear" σ-model, where one starts from a "linear" field ϕ and imposes on it a non-linear constraint, which however <u>does</u> <u>not</u> break the global symmetry under which ϕ used to transform linearly:

$$h \in 4 \quad ; \quad \sum_n h_n h_n = v^2 \; \underbrace{}_{}\!\!\to \text{non-linear constraint}$$

Similarly, we might describe the same theory in terms of the matrix U. Given

$$\phi = h_4 \mathbb{1} + i h_i \sigma^i$$

the non-linear constraint becomes:

$$\phi^+ \phi = (h_4 \mathbb{1} - i h_a \sigma^a)(h_4 \mathbb{1} + i h_b \sigma^b) =$$
$$= h_4^2 \mathbb{1} + (h_a)^2 \mathbb{1} = v^2 \mathbb{1}$$

$$\Rightarrow \quad \phi = U \cdot v \qquad \text{where } U \in SU(2)$$

QUESTION: why $U \in SU(2)$ and not $\in U(2)$?

One parametrizes $U = e^{i\pi_a \sigma^a / v}$ and writes the most general Lagrangian with 2 derivatives compatible with the SO(4) symmetry:

$$U \longrightarrow g_L \, U \, g_R^{\dagger}$$

$$\mathcal{L} = \frac{v^2}{4}\,\mathrm{tr}\left[\partial_\mu U^\dagger \partial^\mu U\right] = \frac{F^2}{16}\,\mathrm{tr}\left[\partial_\mu U^\dagger \partial^\mu U\right]$$