WEINBERG 19.5
GEORGI 2.6

• An Effective Theory of pions • (II) ①

– We have discussed two equivalent forms of the Goldstone Lagrangian:

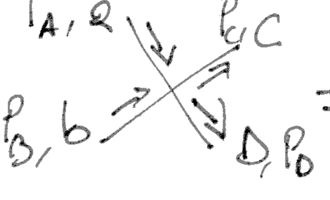
$$\mathcal{L}^{①} = \frac{1}{2}\,\frac{\partial\vec{\pi}\cdot\partial\vec{\pi}}{(1+\vec{\pi}^2/F^2)^2}$$

$$\mathcal{L}^{②} = \frac{F^2}{16}\,\mathrm{tr}\left[\partial_\mu U^+ \partial^\mu U\right];\quad U = e^{2i\pi^a\sigma^a/F}$$

– Looking at $\mathcal{L}^{①,②}$, we explicitely see that many (an ∞ number) of interaction vertices are present, the theory (called "non-linear σ-model") is indeed highly non-linear. The first of such interaction is a 4-point vertex that mediates $\pi\pi \to \pi\pi$ scattering:

$$\mathcal{L}^{①} = \frac{1}{2}\partial\vec{\pi}\cdot\partial\vec{\pi} - \frac{1}{F^2}(\vec{\pi}\cdot\vec{\pi})(\partial\vec{\pi}\cdot\partial\vec{\pi}) + \ldots$$

This interaction is very different from the one you are familiar with, like ϕ^4; its strenght <u>grows</u> with energy. The Feynman rule is indeed:



$$= -\frac{i}{F^2}\cdot 2\cdot 2\left[\delta_{ab}\,\delta_{cD}\left[-p_c\cdot p_D - p_A\cdot p_D\right] + \right.$$

$$+ \delta_{ac}\delta_{bd}\left[P_A \cdot P_C + P_B \cdot P_D\right] + \delta_{ad}\delta_{bc}\left[P_A \cdot P_D + P_B \cdot P_C\right]\Big]$$

If all the scattered particles have a common energy E, the strength of the interaction is E^2/F^2:

$$E/F = \text{"coupling"}$$

E/F plays in this model the <u>same</u> <u>role</u> the e.m. coupling "e" plays in QED. Of course, the difference is that in QED e does not change with E (better said, it changes slowly)

The smaller E, the smaller the coupling, the theory then becomes <u>more</u> <u>and</u> <u>more</u> <u>perturbative</u>; the tree-level results are more and more similar to the exact ones. Remember that for $E \to 0$ our theory becomes equivalent to current algebra, we can then identify

Current Algebra $\Longleftrightarrow$ tree-level in the σ-model

Loops (i.e. E^2/F^2) corrections in the σ-model are effects that go <u>beyond</u> Current Algebra.

Now that we know the $\pi\pi \to \pi\pi$ vertex, we can use it to predict the pion scattering amplitude provided we have measured F. We actually know F already because, as I will show you right now,

$$F = F_\pi = 184\,\text{MeV};$$

where, remember, $\langle 0 | A^\mu_a(0) | \pi_b \rangle = \frac{F_\pi}{2} \delta_{ab}\, p^\mu$

To show this, let us simply compute the matrix element of the current A in the σ-model. To obtain this current it is better to work with the "second form" of the Lagrangian, $\mathcal{L}^{(2)}$. The infinitesimal $SU(2)_L \times SU(2)_R$ transformation acts on U as:

$$U \to g_L U g_R^+ = U + i\varepsilon_L^a \tau^a U - i\varepsilon_R^a U \tau^a$$

So that, applying the Noether theorem I get:

$$\delta\mathcal{L} = \frac{F^2}{16} \mathrm{tr}\Big[\partial_\mu U^+ (i) \partial_\mu \varepsilon_L^a \tau^a U - \partial_\mu U^+ (i) \varepsilon_R^a U \tau^a +$$

$$- i \partial_\mu \varepsilon_L^a U^+ \tau^a \partial_\mu U + i \partial_\mu \varepsilon_R^a \tau^a U^+ \partial_\mu U\Big]$$

$$\Rightarrow J_L^{\mu,a} = i\frac{F^2}{16}\mathrm{tr}\left[\partial_\mu U^+ \tau^a U - U^+\tau^a\partial_\mu U\right]$$

$$= i\frac{F^2}{16}\mathrm{tr}\left[\left(U\partial_\mu U^+ - \partial_\mu U\cdot U^+\right)\tau^a\right]$$

$$= i\frac{F^2}{8}\mathrm{tr}\left[U\partial_\mu U^+\tau^a\right] =$$

$$= i\frac{F^2}{8}(-i)\frac{2}{F}\frac{1}{2}\mathrm{tr}\left[\sigma^b\sigma^a\right]\partial_\mu\pi^b + \dots$$

$$= \frac{F}{4}\partial_\mu\pi^a + \dots$$

And: $J_R^{\mu,a} = +i\frac{F^2}{16}\mathrm{tr}\left[\partial_\mu U\tau^a U^+ - U\tau^a\partial_\mu U^+\right]$

$$= -\frac{F}{4}\partial_\mu\pi^a + \dots$$

For the sake of computing the current matrix element at the tree-level, the first term in the expention in powers of π is all what we need. We obtain

$\langle 0|V_a^\mu|\pi_b\rangle = 0$ Right, it is because of Parity

$$\langle 0|A_a^\mu|\pi_b\rangle = \frac{F}{2}p^\mu\delta_{ab}$$

$\hookrightarrow$ remember there will be an "i" when going from $\mathcal{L}$ to the Feynman rule

The effective field theory method is not only useful to recover known results, it also allows to easily introduce "deformations" of the picture such as, importantly for physical applications in QCD, the <u>explicit</u> breaking of the chiral symmetry that is due, in QCD, to the non-vanishing quarks mass term. Theories in which the explicit breaking is treated as a tiny perturbation are called "Chiral Perturbation Theories", χ_{PT}; in these theories an expansion is performed in the quark mass term i.e., from the perspective of the low-energy effective theory, in the parameter or SPURION, that breaks the chiral symmetry explicitely. We will discuss the most complete version of χ_{PT}, in which 3 flavors of quarks are taken into account, in the following lesson, for the time being let us just discuss the effects of the quark mass in the 2-flavors case.

The "perturbation" is, in the QCD Lagrangian

$$\mathcal{L}_M = -\left[m_u \bar{u}_L u_R + m_d \bar{d}_L d_R\right] + h.c.$$

$$= - (\bar{u}_L, \bar{d}_L) \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \begin{pmatrix} u_R \\ d_R \end{pmatrix} + \text{h.c.}$$

$$= - (\bar{q}_L)_i \, M_{ij} (q_R)_j + \text{h.c.}$$

The presence of these masses obviously breaks the chiral group, if $m_u = m_d$ still $SU(2)_V$ is unbroken, and since $SU(2)_V$ is also preserved by the spontaneous symmetry breaking mechanism, $SU(2)_V$ will be a completely unbroken symmetry both in the spectrum and in the interactions. When $m_u \neq m_d$, $SU(2)_V$ is explicitly broken, and this breaking might show up in both the spectrum and the interactions. It will however not show up at the leading order in the spectrum and interactions of the pions, as we will see.

To construct this χ_{PT} we start from the leading order Lagrangian

$$\mathcal{L}^{(2)} = \frac{F^2}{16} \operatorname{tr}[\partial_\mu U^\dagger \partial^\mu U]$$

The symmetry is exact, and there are 3 massless pions. Turning on M, if it is small, will change the theory only slightly. The relevant degrees of freedom are always the 3 pions, described by the matrix U. The new parameter that will enter in the low-energy theory will be the quark mass-matrix M that breaks the symmetry. Actually, M would not break the symmetry if instead of being made of fixed input parameters it was made by a collection of new degrees of freedom. If this was the case we might simply declare that M transforms as:

$$M_{ij} \longrightarrow (g_L)_{ii'} M_{i'j'} (g_R^+)_{j'j}$$

the mass term would be perfectly invariant if we assign to M these transformation property. For definitness, we might regard M as a field $M(x)$. Of course, it is a completely non-dynamical fields, which does not lead to any physical particle. A "field" of this sort is called a "Spurion". The spurion breaks the symmetry because its physical value (or, its VEV) is not invariant:

$$M = \langle M \rangle = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

But before M taking VEV, we can (most) use the full chiral symmetry to write the Lagrangian. Moreover, since M is a small perturbation, the most relevant term will be the one with less powers of M. Terms with more powers clearly exist, and have to be included order by order. They however lead to small corrections. The leading term of the series has one power of M and it is unique:

$$\mathcal{L}^{(2)}_{MASS} = - \mathcal{C} \Lambda^3_{QCD} \, \mathrm{tr}\left[M^+ U \right] + h.c$$

where $\mathcal{C}$ is an $\mathcal{O}(1)$ coefficient while the prefactor Λ^3_{QCD} has been estimated on dimensional grounds. Notice that $M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$ can be chosen real; then

$$\mathcal{L}^{(2)}_{MASS} = + \mathcal{C} \Lambda^3_{QCD} \, \mathrm{tr}\left[M (U + U^+) \right]$$

But now, remember that

$$U = e^{i \frac{2\pi^a}{F} \sigma^a} = \cos\left[\sqrt{4\pi^2/F^2}\right] \mathbb{1} + i\, 2 \frac{\pi^a}{F} \sigma^a \frac{\sin\left[\sqrt{4\pi^2/F^2}\right]}{\cancel{\sqrt{4\pi^2/F^2}}}$$

$$U + U^+ = 2 \cos\left[\sqrt{4\pi^2/F^2}\right] \cdot \mathbb{1}$$

$$\Rightarrow \mathcal{L}^{(2)}_{MASS} = + 2 \mathcal{C} \Lambda^3_{QCD} \, \mathrm{tr}[M] \cdot \cos\left[2 \sqrt{\pi^2}/F\right]$$

In a totally accidental manner, at the leading order in m, the low energy lagrangian is insensitive to $m_u - m_d$. Even if $m_u \neq m_d$, the pion masses and interactions will still respect the $SU(2)$ isospin. This kind of selection rules is one of the many examples in which a careful use of symmetries (even if broken in all possible ways!) leads to surprising and unexpected results.

By expanding $\mathcal{L}^{(2)}_{MASS}$ we find:

$$\mathcal{L}^{(2)}_{MASS} = 2\mathcal{E}\Lambda^3_{QCD}(m_u+m_d)\left[-\frac{1}{2}\frac{4\vec{\pi}^2}{F^2} + \frac{1}{4!}\frac{16(\vec{\pi}\cdot\vec{\pi})^2}{F^4} + \ldots\right]$$

$$= +\frac{4}{F^2}2\mathcal{E}\Lambda^3_{QCD}(m_u+m_d)\left[-\frac{\vec{\pi}^2}{2} + \frac{1}{6}\frac{(\vec{\pi}\cdot\vec{\pi})^2}{F^2}\right]$$

$$\Rightarrow \quad m_\pi^2 = \frac{8\mathcal{E}}{F^2}\Lambda^3_{QCD}(m_u+m_d)$$

This tells us several things. First, m_π is generated, as expected, from the breaking of the symmetry. Second, the SQUARE pion mass is proportional to the symmetry breaking parameter. So we shouldn't worry so much if m_π is not that small, it is $(m_\pi/\Lambda_{QCD})^2$ that control the corrections, i.e. the departure from the chiral limit. Third, once we fit ε to reproduce the right measured value of m_π, we can predict a new contribution, proportional to m_π^2, to the pion scattering vertex. These corrections to the amplitude have been included, and improve the phenomenological agreement of the model. This is one of the many experimental validations of the entire approach.