• The third Flavor •

It is straight-forward to generalize the discussion of the chiral theories to include the third flavor which is present in the QCD lagrangian. Actually, much more than 3 flavors of quarks exist, but only 3 of them are relevant for the physics of the light ($m \lesssim 1\,GeV$) hadrons. The others are too heavy ($m_q \gtrsim 1\,GeV$) to be relevant. Moreover, their mass is above the confinement scale Λ_{QCD}, treating them in a chiral pert. theory approach is completely unjustified. Other methods of approximation exist and allow to deal with the heavy quarks, these are non-relativistic theories based exactly on the opposite assumptions than the one we are doing here: that they are heavy compared to Λ_{QCD}

Coming back to the ligth flavors, we have 3:

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} ; \quad Q = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix} ; \quad m \simeq \begin{pmatrix} 3\ MeV \\ 6\ MeV \\ 150\ MeV \end{pmatrix}$$

where the values given for the masses are only an estimate; the precise definition and determination of the masses is rather involved.

If the 3 flavors are light, we consider, in analogy to what we did for the pions, a non-linear σ-model with

$$SU(3)_L \times SU(3)_R \to SU(3)_V$$

Instead of an $SU(2)$ matrix, the Goldstones form now an $SU(3)$ matrix U, transforming as:

$$U \to g_L U g_R^+$$

with a Lagrangian:

$$\mathcal{L} = \frac{F^2}{16} \operatorname{tr}\left[\partial_\mu U^+ \partial^\mu U\right]$$

The degrees of freedom in U are 8:

$$U = \exp\left[i \frac{2\sqrt{2}}{F} B\right]$$

$$B \equiv \begin{bmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0 & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta^0 \end{bmatrix}$$

B parametrizes the most general 3×3 hermitian matrix, you could write $B = \sum_a \xi_a \lambda^a$, where λ^a are the usual Gell-Mann matrices.

You might check that the assignement of the electric charge of each component is the correct one if you work out the transformation properties of B under $U(1)_{em}$. Notice that a $U(1)_{em}$ is a vector rotation with generator

$$Q = \begin{pmatrix} 2/3 & & \\ & -1/3 & \\ & & -1/3 \end{pmatrix} = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix} +$$

$$+ \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2\sqrt{3}} \lambda_8 + \frac{1}{2} \lambda_3$$

Under $U(1)_{em}$:

$$U \to e^{i\alpha Q} U e^{-i\alpha Q} \Longrightarrow B \to e^{i\alpha Q} B e^{-i\alpha Q} \simeq$$
$$\cong B + i\alpha [Q, B]$$

By computing $[Q, B]$ and seeing what is the charge q_a of each entry we can check our assignement of the charges: $[Q,B] = \begin{bmatrix} 0 & \pi^+ & K^+ \\ -\pi^- & 0 & 0 \\ -K^- & 0 & 0 \end{bmatrix}$

Notice also that to make the terms involving only pions equal to the ones we already had, we must identify

$$F = F_\pi$$

We do not need, then, to measure any new parameter. Our prediction is that, because of SU(3) chiral, 8 light pseudo-scalar mesons exist and that their interactions are fixed, at the leading order, ~~and~~ are given by our chiral Lagrangian.

As in the 2-flavor case, we can introduce in the Lagrangian the perturbation due to the quark masses:

$$\mathcal{L}_{MASS} = \varepsilon \Lambda^3_{QCD}\, tr\left[M(U+U^+)\right] =$$

$$= \varepsilon \Lambda^3_{QCD}\, tr\left[M\left(1 + i\frac{2\sqrt{2}}{F_\pi}B - \frac{\cancel{4}}{F_\pi^2}B^2 + h.c\right)\right]$$

$$= \varepsilon \Lambda^3_{QCD}\,(-)\frac{16}{2F_\pi^2}\, tr\left[M \cdot B^2\right] =$$

$$= -\frac{1}{2}\,\frac{16\varepsilon\Lambda^3_{QCD}}{F_\pi^2}\left[m_u (B^2)_{11} + m_d (B^2)_{22} + m_s (B^2)_{33}\right]$$

$$(B^2)_{11} = \left(\frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}}\right)^2 + \pi^+\pi^- + K^+K^-$$

$$(B^2)_{22} = \pi^+\pi^- + \left(-\frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}}\right)^2 + K^0\overline{K^0}$$

$$(B^2)_{33} = K^+\overline{K} + K^0\overline{K^0} + \frac{2}{3}(\eta^0)^2$$

The entire mass matrix is predicted in terms of 3 parameters, i.e. the two dimentionless ratios m_d/m_s and m_u/m_s, plus the overall size of the masses, which however is unmeasurable because it appears multiplied by the unknown factor ε. We can however relate the ratios m_d/m_s, m_u/m_s to the observed masses of the Goldstones. After these will be fixed to the observation, we will still have predictions.

Before comparing with observations, there is another effect that we should include: the electro-magnetic corrections to the masses. These can be included, similarly to what we did for the quark masses, with the method of spurions. We add e.m. interactions to the QCD Lagrangian:

$$\mathcal{L}_{em} = e\left[(\bar{q}_L)_i Q^L_{ij} \not{A}\, q_{Lj} + (\bar{q}_R)_i Q^R_{ij} \not{A}\, q_{Rj}\right]$$

where, in reality, $Q^L = Q^R = Q = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$

The spurions $Q^{L,R}$ transform as

$$Q^L \to g_L Q^L g_L^+$$
$$Q^R \longrightarrow g_R Q^R g_R^+$$

The first invariant is at order Q^2:

$$\mathcal{L}^{em} = \Lambda^4 e^2 \operatorname{tr}\left[Q^L U Q^R U^+\right] =$$

$$= \Lambda^4 e^2 \operatorname{tr}\Big[\cancel{Q\, i \frac{2\sqrt{2}}{F_\pi} B Q + Q^2 (-i) \frac{2\sqrt{2}}{F_\pi} B} +$$

$$+ Q(-)\frac{1}{2}\frac{4\cdot 2}{F_\pi^2} B^2 Q + Q^2 (-)\frac{1}{2}\frac{4\cdot 2}{F_\pi^2} B^2 +$$

$$+ Q\, i \frac{2\sqrt{2}}{F_\pi} B\, Q (-) i \frac{2\sqrt{2}}{F_\pi} B\Big] =$$

$$BQQB$$

$$= -\frac{\Lambda^4}{F_\pi^2} e^2\, 4 \operatorname{tr}\Big[QBBQ + \underbrace{QQBB}_{\parallel} + -QBQB - QBQB\Big]$$

$$= -4e^2 \frac{\Lambda^4}{F_\pi^2} \operatorname{tr}\Big[QB[B,Q] - BQ[B,Q]\Big] =$$

$$= -4e^2 \frac{\Lambda^4}{F_\pi^2} \operatorname{tr}\left[\left([B,Q]\right)^2\right] = \quad \rightarrow \text{we computed before } [B,Q]$$

$$= -4e^2 \frac{\Lambda^4}{F_\pi^2}\left[\pi^+\pi^- + K^+K^-\right]$$

We observe an interesting fact: em corrections can only affect, at the leading (e^2) order, the <u>charged</u> Goldstones mass. Moreover, they equally affect pions and Kaons

Putting all these informations toghether we obtain predictions on the spectrum. The reader is referred to Weinberg pg 230-231 for more details (in this notation, the electromagnetic correction to π and K masses is denoted as "Δ". Let us just summarize the most relevant results:

- One relation called "Gell-Mann-Okubo".

$$3m_\eta^2 + 2m_{\pi^\pm}^2 - m_{\pi^0}^2 = 2m_{K^\pm}^2 + 2m_{K^0}^2$$

- Predictions for the mass ratios:

$$m_d/m_s = 0.050 \quad ; \quad m_u/m_s = 0.027$$