• One-Loop Renormalization of the scalar theory with shift simmetry •

The theory we are interested in, as discussed in the previous lecture, has a Lagrangian

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu \phi)(\partial^\mu \phi) + \sum_i \frac{g(n_i,d_i)}{M^{n_i+d_i-4}}\,\mathcal{O}^{(n_i,d_i)}$$

Let us first of all classify the first few allowed operators. Remember that the main point will be of ensuring that they are compatible with the symmetries:

$$\phi \longrightarrow -\phi \qquad \text{and} \qquad \phi \longrightarrow \phi + \varepsilon$$

Start classifying by the number of fields n_i:

$\underline{n_i = 2}$: I must have 2-derivatives or more:

$d_i = 2$: $(\partial\phi)^2$

$d_i = 3$: impossible for Lorentz

$d_i = 4$: $\phi \cdot \Box^2 \phi$ because by integration by parts I can always put all the ∂ on the last ϕ

Actually, we are not obliged to put the $\phi \Box^2 \phi$ term. Remember indeed that the physical content of our theory does not depend on the field basis in which you write the Lagrangian. There is even a Theorem saying that two Lagrangians related by field redefinition are equivalent: they give identical S-matrix elements. We already commented on this when talking about the linear and non-linear field basis for the SO(4) σ-model. The $\phi \Box^2 \phi$ term can be removed by field redefinition: suppose I put it in:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) + g(4,2) \frac{1}{M^2} (\Box^2 \phi) \phi + \cdots$$

Redefine: $\phi = \phi' + A \Box \phi'$

$$\mathcal{L} = \frac{1}{2} (\partial \phi')^2 - A \phi' \Box \cdot \Box \phi' + g(4,2)/M^2 \, \phi' \Box^2 \phi'$$
$$+ \frac{1}{2} A^2 (\partial \Box \phi')^2 + \cdots$$

Of course I will introduce other operators, that were however already present, but by choosing A we can cancel the $g(4,2)$ term.

Actually, all the $d_L=2$ terms are of the form $\phi \Box^n \phi$, and they can all be removed by a field redefinition. Actually, all the terms that contain $\Box\phi$ can be removed:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \alpha\, \Box\phi \cdot \mathcal{O}[\phi,\partial]$$

$$\downarrow \quad \phi \to \phi + \beta\, \mathcal{O}[\phi,\partial]$$

$$\frac{1}{2}(\partial\phi)^2 - \beta\, \Box\phi\, \mathcal{O}[\phi,\partial] + \alpha \Box\phi\, \mathcal{O}[\phi,\partial]$$

$$+ \dots$$

the linear term in β can cancel the operator

With this new rule, it becomes very simple to continue the classification:

$n_L = 4$: $\quad d_L = 4$: $(\partial_\mu\phi\, \partial^\mu\phi)^2$

$d_L = 6$: $\partial_\mu\phi\, \partial_\nu\phi\, \partial_\rho\phi\, \partial_\sigma\phi$,

$\partial_\mu\partial_\lambda\phi\, \partial^\mu\partial^\lambda\phi\, \partial_\rho\phi\,\partial^\rho\phi \leftarrow$ 2.doc

We could also have a term:

$$\partial_\mu \partial_\lambda \phi \, \partial^\mu \phi \, \partial^\nu \phi \, \partial_\nu \partial^\lambda \phi =$$

$$= -\partial_\mu \phi \, \partial^\mu \partial_\lambda \phi \, \partial^\nu \phi \, \partial_\nu \partial^\lambda \phi - \partial_\mu \phi \, \partial^\mu \phi \, \partial^\nu \partial^\lambda \phi \, \partial_\nu \partial_\lambda \phi +$$

integrating by parts

"Box term"

$$\Rightarrow \quad 2 \partial_\mu \partial_\lambda \phi \, \partial^\mu \phi \, \partial^\nu \phi \, \partial_\nu \partial^\lambda \phi = -(\partial \phi)^2 \partial_\mu \partial_\nu \phi \, \partial^\mu \partial^\nu \phi + \text{"box"}$$

It can therefore be rewritten as the previous one.

Finally, for the calculation we will make we also need:

$n_\ell = 4$; $d_\ell = 8$:

$$\partial_\mu \partial_\nu \phi \; \partial_\rho \partial_\sigma \phi \; \partial_\lambda \partial_\delta \phi \; \partial_\omega \partial_\eta \phi$$

or

$$\partial_\mu \partial_\nu \partial_\rho \partial_\sigma \phi \; \partial_\delta \partial_\lambda \phi \; \partial_\omega \phi \; \partial_\eta \phi$$

But this unavoidably leads to a $\Box$

$$\partial_\mu \partial_\nu \partial_\rho \phi \, \partial_\lambda \partial_\sigma \partial_\omega \phi \, \partial_\eta \phi \, \partial_\delta \phi$$

But by derivatives by parts this goes to the previous class.

In summary, the operators are

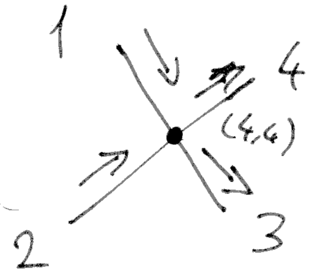
$$\frac{g(4,4)}{M^4} \left(\partial_\mu \phi \partial^\mu \phi\right)\left(\partial_\nu \phi \partial^\nu \phi\right) +$$

$$+ \frac{g(4,6)}{M^6} \left(\partial_\mu \partial_\lambda \phi \, \partial^\mu \partial^\lambda \phi\right)\left(\partial_\rho \phi \partial^\rho \phi\right) +$$

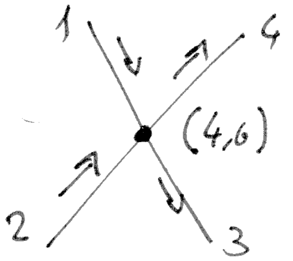
$$+ \frac{g_1(4,8)}{M^8} \partial_\mu \partial_\nu \phi \, \partial^\nu \partial^\lambda \phi \, \partial_\lambda \partial_\rho \phi \, \partial^\rho \partial^\mu \phi +$$

$$+ \frac{g_2(4,8)}{M^8} \left(\partial_\mu \partial_\nu \phi \, \partial^\mu \partial^\nu \phi\right)^2$$

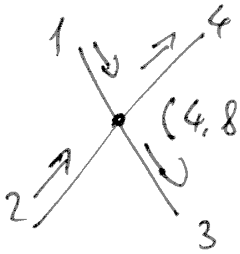
Let us first of all derive their Feynman Rules (you will do this as an exercise)



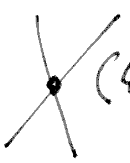
$$= i \frac{8}{M^4} g(4,4) \Big[(P_1 \cdot P_2)(P_3 \cdot P_4) + (P_1 \cdot P_4)(P_2 \cdot P_3) + (P_1 \cdot P_3)(P_2 \cdot P_4) \Big]$$



$$(4,6) = -\frac{4e}{M^6}\, g(4,6) \times \Big[(p_1\cdot p_2)^2 (p_3\cdot p_4) + (p_1\cdot p_3)^2 (p_2\cdot p_4) + (p_1\cdot p_4)^2 (p_2\cdot p_3) + (p_2\cdot p_3)^2 (p_1\cdot p_4) + (p_2\cdot p_4)^2 (p_1\cdot p_3) + (p_3\cdot p_4)^2 (p_1\cdot p_2) \Big]$$



$$(4,8)_1 = \frac{g_1(4,8)}{M^8} \cdot \cancel{4}\, 8 e \times \Big[(p_1\cdot p_2)(p_2\cdot p_3)(p_3\cdot p_4)(p_4\cdot p_1)$$
$$+ (p_1\cdot p_2)(p_2\cdot p_4)(p_4\cdot p_3)(p_3\cdot p_1) +$$
$$+ (p_1\cdot p_3)(p_3\cdot p_2)(p_2\cdot p_4)(p_4\cdot p_1)$$
$$+ (p_1\cdot p_3)(p_3\cdot p_4)(p_4\cdot p_2)\cancel{(p_2\cdot p_1)} +$$
$$+ (p_1\cdot p_4)\cancel{(p_4\cdot p_2)}(p_2\cdot p_3)(p_3\cdot p_1)$$
$$\cancel{+ (p_1\cdot p_4)}(p_4\cdot p_3)(p_3\cdot p_2)(p_2\cdot p_1) \Big]$$



$$(4,8)_2 = \frac{g_2(4,8)}{M^8} \cdot 8e \times \Big[(p_1\cdot p_2)^2 (p_3\cdot p_4)^2 + (p_1\cdot p_3)^2 (p_2\cdot p_4) + (p_1\cdot p_4)^2 (p_2\cdot p_3)^2 \Big]$$

Our goal will be to compute the $2 \to 2$ scattering amplitude among the "ϕ" particles, this calculation is quite easier than the generic off-shell 4-point 1PI because the external momenta are on-shell. Notice also that it is only when computing physical (on-shell) amplitudes that the operators we got rid of by EOM are really immaterial. The equivalence of theories obtained by field redefinitions obviously only holds for physical amplitudes, not for the Green's functions.

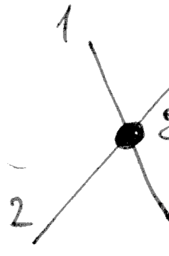
We have discussed in the previous lecture that the leading order contribution to the amplitude comes from $g(4,4)$ at tree-level. To compute it, remember the definition of s, t, u. For massless particles:

$$s = (P_1+P_2)^2 = (P_3+P_4)^2 = 2P_1 \cdot P_2 = 2P_3 \cdot P_4$$

$$t = (P_1-P_3)^2 = (P_4-P_2)^2 = -2P_1 \cdot P_3 = -2P_2 \cdot P_4$$

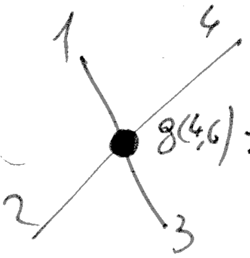
$$u = (P_1-P_4)^2 = (P_3-P_2)^2 = -2P_2 \cdot P_3 = -2P_1 \cdot P_4$$

$$s+t+u = 2P_1 \cdot (P_2 - P_3 - P_4) = -2P_1^2 = 0$$



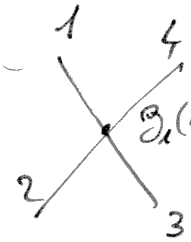
$$g(4,4) = 2i\,\frac{g(4,4)}{M^4}\left[s^2+u^2+t^2\right]$$

This is the leading E^4/M^4, the Next-to-leading (NLO) is E^6/M^6, and comes at tree-level from $g(4,6)$:

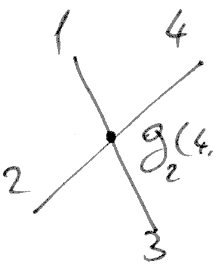


$$g(4,6) = -\frac{i}{\cancel{2}}\,\frac{g(4,6)}{M^6}\left[s^3+t^3+u^3\right]\times\cancel{2}$$

The NNLO, E^8/M^8, comes from the $g_{1,2}(4,8)$ at tree-level, and from the Loop made with $g(4,4)$. Let us first compute the tree-level effects:



$$g_1(4,8) = \frac{i}{2}\,\frac{g_1(4,8)}{M^8}\left[s^2u^2+s^2t^2+t^2u^2\right]$$

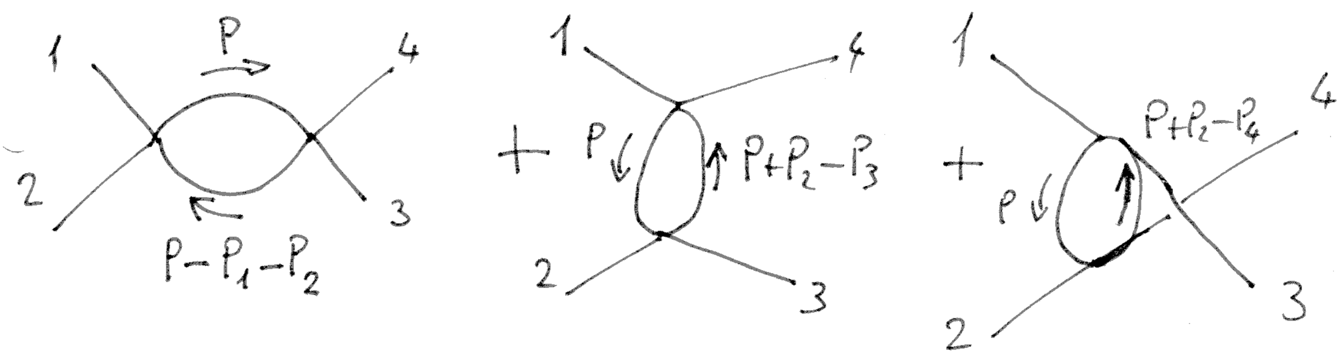


$$g_2(4,8) = \frac{i}{2}\,\frac{g_2(4,8)}{M^8}\left[s^4+t^4+u^4\right]$$

We should now compute the loop. We will not actually perform the calculation, our main goal is to show that the cutoff-dependent part of the integral can be reabsorbed in the redefinition of the couplings. The couplings that will have to be redefined (renormalized) are $g(4,4)$, $g(4,6)$, $g_{1,2}(4,8)$.

Before starting, let us discuss a simpler example: the $\lambda\phi^4$ theory:

$$\times = i\lambda \qquad\qquad \text{———} = \frac{i}{p^2-m^2+i\varepsilon}$$



$$= +\lambda^2 \int \frac{d^4p}{(2\pi)^4}\, \frac{1}{p^2-m^2+i\varepsilon}\, \frac{1}{p^2-2p\cdot(p_1+p_2)+s-m^2} + \underset{s\leftrightarrow u}{\overset{\text{CROSSING}}{4\leftrightarrow 2}} + \underset{s\leftrightarrow t}{3\leftrightarrow 2}$$

The integral is regulated by cutoff Λ in the Euclidean momentum.

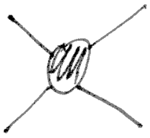
To study the dependence on Λ of the integral, and in particular to identify the divergent terms, just rewrite:

$$\frac{1}{p^2+2p\cdot q+q^2-m^2} = \frac{1}{p^2-m^2}\left[\frac{1}{1+\frac{q^2+2p\cdot q}{p^2-m^2}}\right] =$$

$$= \frac{1}{p^2-m^2}\left[1-\frac{q^2+2p\cdot q}{p^2-m^2}+\left[\frac{q^2+2p\cdot q}{p^2-m^2}\right]^2+R_3[p,q]\right]$$

the "residual" R_3 scales as $\sim[q/p]^3$ for $q \ll p$, I could of course continue with R_n. Whatever the degree of divergence of the original integral was, a big enough "n" can be found such that the convergence of the integral is improved enough to become <u>finite</u>. With this substitution, the divergent terms can be isolated from the rest. These terms are a <u>polynomial</u> in the <u>external</u> <u>momenta</u>. This is definitely <u>general</u>, for any amplitude in any QFT, you could always perform this trick of expanding the denominators.

For the $\lambda \phi^4$, one subtraction is enough ($R_n = R_1$):



$$= i\lambda + \lambda^2 \int \frac{d^4 p}{(2\pi)^4} \left(\frac{1}{p^2 - m^2} \right)^2 + \text{"finite"} + \text{"}s \leftrightarrow t\text{"} + \text{"}s \leftrightarrow u\text{"}$$

$$= i\lambda + i\lambda^2 3 \cdot \int \frac{d\Omega_3}{(2\pi)^4} \int_0^{\Lambda} \frac{p_E^3 \, dp_E}{(p_E^2 + m^2)^2} + \text{"finite"}$$

$$= i\lambda + i 3 \lambda^2 \frac{2\pi^2}{16\pi^4} \times \frac{1}{2} \left[\log\left[1 + \frac{\Lambda^2}{m^2}\right] - \frac{\Lambda^2}{\Lambda^2 + m^2} \right] + \text{"finite"}$$

$$= i\lambda + i 3 \lambda^2 \frac{1}{16\pi^2} \log \frac{\Lambda^2}{m^2} - i 3 \frac{\lambda^2}{16\pi^2} + \text{"finite"}$$

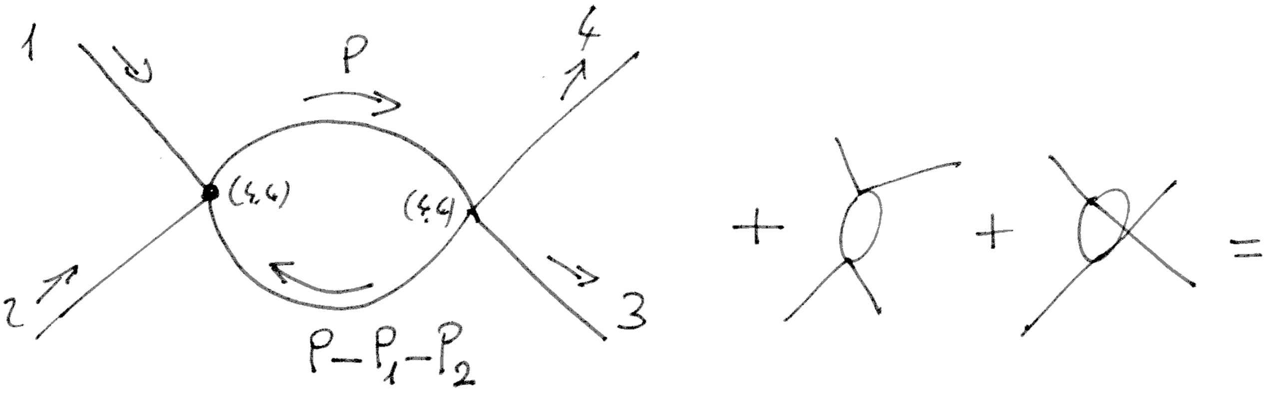
If you want, compute also the finite part as an exercise. From this example we see that:

1) The divergence can be reabsorbed (hidden) in λ because it has <u>local structure</u>

2) To each loop it is associated a factor of $1/16\pi^2$. It shows up in both the divergent and finite parts.

3) The mechanism easily generalizes to more divergent loop integrals.

Let us go back to our problem, you will check in an exercise that the one-loop amplitude reads:



$$= + \frac{64\,(g(4,4))^2}{M^8} \int \frac{d^4P}{(2\pi)^4} \frac{1}{P^2+i\varepsilon} \frac{1}{(P-P_1-P_2)^2+i\varepsilon} \times$$

$$\left[-\frac{s}{2} P^2 - 2\,(P\cdot P_1)(P\cdot P_2) + s\left[P\cdot(P_1+P_2) \right] \right] \times$$

$$\left[-\frac{s}{2} P^2 - 2\,(P\cdot P_3)(P\cdot P_4) + s\left[P(P_3+P_4) \right] \right] +$$

$$+ \text{"crossings"}$$

By inspection, the leading divergence is $\int d^4P \sim \Lambda^4$, we would need "$n=5$" in the expantion of the numerator to make the residual finite:

$$\frac{1}{(P+q)^2} = \frac{1}{P^2} \cdot \left[1 - \frac{q^2+2P\cdot q}{P^2} + \left[\frac{q^2+2P\cdot q}{P^2} \right]^2 + \cdots + R_5 \right]$$

The various divergences will be reabsorbed as follows: (13)

Λ^4 : renormalizes $g(4,4)$

Λ^3 : not there ($\int d^4p\, p^{2n+1} = 0$)

Λ^2 : renormalize $g(4,6)$

Λ : not there

log : renormalize $g_{1,2}(4,8)$

Let us start from the leading Λ^4:

$$\frac{64\,(g(4,4))^2}{M^8} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2} \frac{1}{p^2} \times \frac{1}{4} \times$$

$$\left[s\,p^2 + 4(p\cdot p_1)(p\cdot p_2) \right] \left[s\,p^2 + 4(p\cdot p_3)(p\cdot p_4) \right]$$

where only the terms that lead to Λ^4 have been retained. In order to further simplify this one needs two identities, that follow from Lorentz invariance

$$\int d^4p\, p_\mu p_\nu\, f(p^2) = \frac{1}{4}\, \eta_{\mu\nu} \int p^2 f(p^2)\, d^4p$$

$$\int d^4p \; p_\mu p_\nu p_\rho p_\sigma \, f(p^2) = \frac{1}{24}\left[\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\right] \times \int d^4p \; p^4 f(p^2)$$

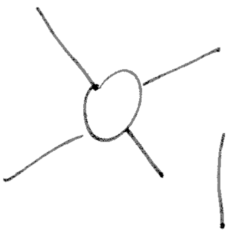
You can check that one gets:

$$\frac{64\,(g(4,4))^2}{M^8}\,\frac{1}{4}\,2\left[s^2 + \frac{4}{3}(s^2+t^2+u^2)\right]\int \frac{d^4p}{(2\pi)^4}$$

+ "crossing"

$$\int \frac{d^4p}{(2\pi)^4} = \overset{\text{from Euclidean}}{i}\,\frac{1}{16\pi^2}\,\frac{\Lambda^4}{2}$$

the crossing are two more terms, obtained from the latter by $s \leftrightarrow t$ and $s \leftrightarrow u$:



$$\Big|_{Div(\Lambda^4)} = \frac{16i\,(g(4,4))^2}{M^8}\,\frac{\Lambda^4}{16\pi^2}\,5\left[s^2+t^2+u^2\right]$$

this has the same form of $g(4,4)$, so it renormalizes the latter.

Let us now look at the subleading divergence. That the Λ^3 term cannot be there comes from the fact that

$$\int d^4p \, \frac{p^\mu}{p^2} = 0$$

by Lorentz Symmetry (which is respected by our regulator).

The Λ^2 term come from several places. First of all from picking the subleading terms in the numerator, still keeping the leading one in the denominator:

$$\frac{64\,(g(4,4))^2}{M^8} \int \frac{d^4p}{(2\pi)^4} \, \frac{1}{p^4} \, s^2 \left[p(p_1+p_2)\right]\left[p(p_3+p_4)\right]$$

$$= \frac{64\,(g(4,4))^2}{M^8} \int \frac{d^4p}{(2\pi)^4} \, \frac{1}{p^{\cancel{4}2}} \, \frac{1}{4} \, p^2 \cdot s^3 \; + \text{"crossing"}$$

this renormalizes $g(4,6)$!

Other terms come from the denominator

$$\frac{1}{(p+q)^2} = \frac{1}{p^2}\left[1 - \underbrace{\frac{2p\cdot q}{p^2}}_{\text{this one combines with the } p^3 \text{ in the numerator}} - \frac{q^2}{p^2} + 4\,\frac{(p\cdot q)^2}{p^4} + \overbrace{\frac{q^4}{p^4}}^{\text{this only contributes to } \log\Lambda} + \dots\right]$$

It becomes more and more tedious to go on, but you might continue and show that the quadratic div. is reabsorbed in $g(4,6)$. Needless to say, the tasks become trivial if using computer algebra manipulators. One could get until the end, and after canceling the divergences compute the finite part, the one after renormalization. We would have then computed the $2 \to 2$ scattering at one loop in our scalar theory. In spite of the fact that it was not renormalizable.