Eidgenössische Technische Hochschule Zürich
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Quantum Information Theory
Series 6

FS 11
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Exercise 6.1 Distance bounds

In this exercise we'll bring together some of the concepts and techniques you have been learning over the past three weeks: purification, fidelity and trace distance. This result is very useful; don't forget to ask your TA why!

a) Show that any purification of the state $\rho_{AB} = \frac{\mathbb{1}_A}{|\mathcal{H}_A|} \otimes \rho_B$ has the form

$$|\psi\rangle_{AA'BB'} = |\Psi\rangle_{AA'} \otimes |\psi\rangle_{BB'},$$

where $|\Psi\rangle_{AA'} = |\mathcal{H}_A|^{-\frac{1}{2}} \sum_i |i\rangle_A |i\rangle_{A'}$ is a maximally entangled state, and $|\psi\rangle_{BB'}$ is a purification of ρ_B.

b) Consider a state that is ε-distant from ρ_{AB} according to the trace distance, i.e.

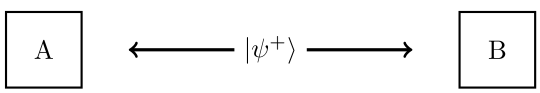
$$\delta\left(\sigma_{AB}, \frac{\mathbb{1}_A}{|\mathcal{H}_A|} \otimes \rho_B\right) \leq \varepsilon.$$

Find an upper bound for

$$\delta\left(|\phi\rangle_{ABP}, |\Psi\rangle_{AA'} \otimes |\psi\rangle_{BB'}\right),$$

where $|\phi\rangle_{ABP}$ is a purification of σ_{AB}. You can take for instance $\mathcal{H}_P = \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$.

Exercise 6.2 Bell-type Experiment



Consider a 2-qubit Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ with basis $\{|00\rangle, |01\rangle |10\rangle, |11\rangle\}$ in the Bell-state

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right). \tag{1}$$

Two parties, called Alice and Bob, get half of the state $|\psi^+\rangle$ so that Alice has qubit A and Bob has qubit B. Alice then performs a measurement $\mathcal{M}_A^\alpha := \{|\alpha\rangle\langle\alpha|, |\alpha\rangle^\perp\langle\alpha|^\perp\}$, with $|\alpha\rangle := \cos(\alpha)|0\rangle + \sin(\alpha)|1\rangle$, on her qubit.

a) Find the description Bob would give to his partial state on B after he knows that Alice performed the measurement M_A^α on A. What description would Alice give to ρ_B given that she knows what measurement outcome she received?

b) If Bob does the measurement $\mathcal{M}_B^0 = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ on B, what is the probability distribution for his outcomes, $\Pr_B$? How would Alice describe his probability distribution, $\Pr_{B|A}$?

c) In part a) and b) Alice and Bob have different descriptions of the quantum state ρ_B and probability distribution of measurement outcomes on that state. Explain how this subjective assignment of the scenarios at B does not contradict with the actual measurement outcomes Bob will get after doing the measurement $\mathcal{M}_B^0$.

Exercise 6.3 Depolarizing channel

We are given two two-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ and a completely positive trace preserving (CPTP) map $\mathcal{E}_p : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ defined as

$$\mathcal{E}_p(\rho) = p\frac{\mathbb{1}}{2} + (1-p)\rho. \tag{2}$$

a) An operator-sum representation (also called the Kraus-operator representation) of a CPTP map $\mathcal{E} : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ is a decomposition $\{E_k\}_k$ of operators $E_k \in \mathrm{Hom}(\mathcal{H}_A \to \mathcal{H}_B)$, $\sum_k E_k {E_k}^* = \mathbb{1}$, such that

$$\mathcal{E}(\rho) = \sum_k E_k \rho {E_k}^*.$$

Find an operator-sum representation for $\mathcal{E}_p$.

Hint: Remember that $\rho \in \mathcal{S}(\mathcal{H}_A)$ can be written in the Bloch sphere representation:

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}), \quad \vec{r} \in \mathbb{R}^3, \quad |\vec{r}| \leq 1, \quad \vec{r}\cdot\vec{\sigma} = r_x\sigma_x + r_y\sigma_y + r_z\sigma_z, \tag{3}$$

where σ_x, σ_y and σ_z are Pauli matrices. It may be useful to show that

$$\mathbb{1} = \frac{1}{2}(\rho + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z).$$

b) What happens to the radius $\vec{r}$ when we apply $\mathcal{E}_p$? How can this be interpreted?

c) A probability distribution $P_A(0) = q$, $P_A(1) = 1-q$ can be encoded in a quantum state on $\mathcal{H}_A$ as $\hat{\rho} = q|0\rangle\langle 0|_A + (1-q)|1\rangle\langle 1|_A$. Calculate $\mathcal{E}(\hat{\rho})$ and the conditional probabilities $P_{B|A}$ as well as P_B after measuring $\mathcal{E}(\hat{\rho})$ in the standard basis $\{|0\rangle, |1\rangle\}$.

d) Maximise the mutual information over q to find the classical channel capacity of the depolarizing channel.