Quantum Information Theory
Series 9

Definitions: von Neumann entropy

In this series we will derive some useful properties of the von Neumann entropy, the quantum version of Shannon entropy. We will also look at the strangeness of quantum mutual information. Before we start, here are a few definitions.
The von Neumann entropy of a density operator $\rho \in \mathcal{S}(\mathcal{H}_A)$ is defined as

$$H(A)_\rho = -\mathrm{Tr}\big(\rho \log \rho\big) = -\sum_i \lambda_i \log \lambda_i, \tag{1}$$

where $\{\lambda_i\}_i$ are the eigenvalues of ρ.
Given a composite system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ we write $H(AB)_\rho$ to denote the entropy of the reduced state of a subsystem, $\rho_{AB} = \mathrm{Tr}_C(\rho_{ABC})$. When the state ρ is obvious from the context we can drop the index.
The *conditional* von Neumann entropy may be defined as

$$H(A|B)_\rho = H(AB)_\rho - H(B)_\rho. \tag{2}$$

In the Alice-and-Bob picture this quantifies the uncertainty that Bob, who holds part of a quantum state, ρ_B, still has about Alice's state.
The strong sub-additivity property of the von Neumann entropy shows up a lot. It applies to a tripartite composite system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$,

$$H(A|BC)_\rho \leq H(A|B)_\rho. \tag{3}$$

Exercise 9.1 Some properties of von Neumann entropy

a) Prove the following general properties of the von Neumann entropy:

1. If ρ_{AB} is pure, then $H(A)_\rho = H(B)_\rho$.
2. If two systems are independent, $\rho_{AB} = \rho_A \otimes \rho_B$, then $H(AB)_\rho = H(A)_{\rho_A} + H(B)_{\rho_B}$.

b) Consider a bipartite state that is classical on subsystem Z: $\rho_{ZA} = \sum_z p_z |z\rangle\langle z|_Z \otimes \rho_A^z$ for some basis $\{|z\rangle_Z\}_z$ of $\mathcal{H}_Z$. Show that:

1. The conditional entropy of the quantum part, A, given the classical information Z is

$$H(A|Z)_\rho = \sum_z p_z\, H(A|Z=z), \tag{4}$$

where $H(A|Z=z) = H(A)_{\rho_A^z}$.

2. The entropy of A is concave,

$$H(A)_\rho \geq \sum_z p_z\, H(A|Z=z). \tag{5}$$

3. The entropy of a classical probability distribution $\{p_z\}_z$ cannot be negative, even if one has access to extra quantum information, A,

$$H(Z|A)_\rho \geq 0. \tag{6}$$

Remark: Eq (6) holds in general only for classical Z. Bell states are immediate counterexamples in the fully quantum case.

Exercise 9.2 Upper bound on von Neumann entropy

Given a state $\rho \in \mathcal{S}(\mathcal{H}_A)$, show that

$$H(A)_\rho \leq \log |\mathcal{H}_A|. \tag{7}$$

Hints: Consider the state $\bar{\rho} = \int U \rho U^\dagger dU$, where the integral is over all unitaries $U \in \mathcal{U}(\mathcal{H})$ and dU is the Haar measure. Find $\bar{\rho}$ and use concavity (5) to show (7).
The Haar measure satisfies $d(UV) = d(VU) = dU$, where $V \in \mathcal{U}(\mathcal{H})$ is any unitary.

Exercise 9.3 Quantum mutual information

One way of quantifying correlations between two systems A and B is through their *mutual information*, defined as

$$I(A:B)_\rho = H(A)_\rho + H(B)_\rho - H(AB)_\rho \tag{8}$$
$$= H(A)_\rho - H(A|B)_\rho. \tag{9}$$

We can also define a conditional version of the mutual information between A and B as

$$I(A:B|C)_\rho = H(A|C)_\rho + H(B|C)_\rho - H(AB|C)_\rho \tag{10}$$
$$= H(A|C)_\rho - H(A|BC)_\rho. \tag{11}$$

a) Consider two qubits A and B in joint state ρ_{AB}.

1. Prove that the mutual information of the Bell state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is maximal. This is why we say Bell states are *maximally entangled*.
2. Show that $I(A:B) \leq 1$ for classically correlated states, $\rho_{AB} = p|0\rangle\langle 0|_A \otimes \sigma_B^0 + (1-p)|1\rangle\langle 1|_A \otimes \sigma_B^1$ (where $0 \leq p \leq 1$).

b) Consider the so-called *cat state* of four qubits, $A \otimes B \otimes C \otimes D$, that is defined as

$$|☺\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle). \tag{12}$$

Check how the mutual information between A and B changes with the knowledge of the remaining qubits,

1. $I(A:B) = 1$.
2. $I(A:B|C) = 0$.
3. $I(A:B|CD) = 1$.