Locality and nonclassicality of quantum theory

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Quantum Information Theory FS11

I have two questions about the topic "quantum nonclassicality" as discussed in the script. On p. 51, the proof that QM is in general not classically local is mentioned (i.e. proof of Lemma 5.3.1). However, I dont get a few things:

1. *In Eq (5.4) we got for the* $P_{xy,ab} = \frac{1}{2}\sin^2(a-b)$ *for* $X \neq Y$*. However, in the proof, we say that* $P[X_{\pi/2} \neq Y_0] = 1$*. Is this probability, i.e. the prob. that the measurement outcome at A and B are not equal for those angles the same as* Pxy, ab *for* $X \neq Y$*? In this case I could not explain where we got the 1 from, I think it should be just a* $1/2$*?*
2. *It is said the the whole proof is based on Lemma 5.2.3. We then assume by contradiction that the probability distributions describing the QM measurement outcome are classically local, i.e.* $P_{XY|ab} = P_{X_a,Y_b}$*. What I did not get here is what this statement implies for our proof. Has this something to do that we can write the joint prob. distr.* $P_{xy,ab}$ *in a form* $P[X_{\pi/2} \neq Y_0]$ *etc.?*

We have two random variables: X, with alphabet $\{x\}_x$, and Y, with alphabet $\{y\}_y$. Their joint probability distribution happens to depend on two parameters, a and b. We denote the probability of obtaining the 'outcome' (x, y) conditioned on the parameters a and b by $P_{XY|ab}(x, y)$.

To assume that this probability distribution is local means that we think that a only concerns X and b only affects Y. Formally, this implies that for each possible value of a there exists a random variable X_a with alphabet $\{x_a\}_x$ (the same for b) such that

$$P_{XY|ab}(x, y) = P_{X_a Y_b}(x_a, y_b), \tag{1}$$

i.e., we are no longer talking about how the random variables XY behave together conditioned on the parameters a and b, but about the joint probability distribution of random variables X_a and Y_b (which are not conditioned on anything).

There are situations where locality clearly does not make sense: for instance, if both X and Y represent experiments that depend on the two parameters, a and b (like 'Alice and Bob should both check if it's raining on date a at time b'). In the case we are considering, it seems logical that the probability distribution is local, because a and b are the local angles that define the bases in which Alice and Bob measure their qubits. However, if we assume locality, we obtain a direct contradiction with the results predicted by quantum mechanics (and vastly confirmed by experiments).

You start from the joint probability distribution of the outcomes Alice and Bob can obtain for two different choices of basis by Alice,

$$P_{XY|\frac{\pi}{2},0}(x,y) = \begin{cases} \frac{1}{2}, & x \neq y; \\ 0, & x = y. \end{cases} = \begin{cases} 0, & (0,0); \\ \frac{1}{2}, & (0,1); \\ \frac{1}{2}, & (1,0); \\ 0, & (1,1). \end{cases} \tag{2}$$

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If you assume locality, then we have

$$P_{X_{\frac{\pi}{2}}Y_0}(x_{\frac{\pi}{2}},y_0) = \begin{cases} 0, & (0,0); \\ \frac{1}{2}, & (0,1); \\ \frac{1}{2}, & (1,0); \\ 0, & (1,1). \end{cases} \qquad P_{X_0Y_0}(x_0,y_0) = \begin{cases} \frac{1}{2}, & (0,0); \\ 0, & (0,1); \\ 0, & (1,0); \\ \frac{1}{2}, & (1,1). \end{cases} \tag{4}$$

Because we assume $X_{\frac{\pi}{2}}$ and Y_0 (as well as X_0 and Y_0) to be two different random variables, we can write

$$P[X_{\frac{\pi}{2}} \neq Y_0] = \sum_{x_{\frac{\pi}{2}} \neq y_0} P_{X_{\frac{\pi}{2}}Y_0}(x_{\frac{\pi}{2}},y_0) = \frac{1}{2} + \frac{1}{2} = 1, \tag{5}$$

$$P[X_0 = Y_0] = \sum_{x_0 = y_0} P_{X_0Y_0}(x_0,y_0) = \frac{1}{2} + \frac{1}{2} = 1. \tag{6}$$

However, it would not make sense to write something as $P[X_0 = Y_0]$ if we did not have the assumption of locality, because in that case random X_0 and Y_0 would not even exist. We could only talk of 'the probability that X equals Y given the parameters $a = \frac{\pi}{2}, b = 0$', denoted by $P[X = Y|0,0]$.

Assuming locality, we could conlude that $P[X_{\frac{\pi}{2}} \neq X_0] = 1$. We have two probability distributions: $P_{X_{\frac{\pi}{2}}Y_0}$, with alphabet $\{(x_{\frac{\pi}{2}},y_0)\}$, and $P_{X_0Y_0}$, with alphabet $\{(x_0,y_0)\}$. If we abstract from the context (that X is Alice and Y is Bob), we can think of them as marginals of the joint probability distribution $P_{X_{\frac{\pi}{2}}X_0Y_0}$, which would have alphabet $\{(x_{\frac{\pi}{2}},x_0,y_0)\}$. In fact, there is one probability distribution that has those two as marginals[1],

$$P_{X_{\frac{\pi}{2}}X_0Y_0}(x_{\frac{\pi}{2}},x_0,y_0)=\begin{cases}0, & (0,0,0);\\ 0, & (0,0,1);\\ \frac{1}{2}, & (0,1,0);\\ 0, & (0,1,1);\\ 0, & (1,0,0);\\ \frac{1}{2}, & (1,0,1);\\ 0, & (1,1,0);\\ 0, & (1,1,1).\end{cases} \tag{7}$$

The marginal on $X_{\frac{\pi}{2}}X_0$ of this probability distribution gives us

$$P_{X_{\frac{\pi}{2}}X_0}(x_{\frac{\pi}{2}},x_0)=\begin{cases}0, & (0,0);\\ \frac{1}{2}, & (0,1);\\ \frac{1}{2}, & (1,0);\\ 0, & (1,1).\end{cases}=\begin{cases}\frac{1}{2}, & x_{\frac{\pi}{2}}\neq x_0;\\ 0, & x_{\frac{\pi}{2}}=x_0.\end{cases} \tag{8}$$

Now we can compute

$$P[X_{\frac{\pi}{2}}\neq X_0]=\sum_{x_{\frac{\pi}{2}}\neq x_0}P_{X_{\frac{\pi}{2}}X_0}(x_{\frac{\pi}{2}},x_0)=\frac{1}{2}+\frac{1}{2}=1. \tag{9}$$

However, the predictions from quantum mechanics (rest of lemma 5.3.1 of the script) give us $P[X_{\frac{\pi}{2}}\neq X_0]<1$. We have to concede that the original joint probability distributions could not have been local.

[1] which could be found by solving the system of linear equations

$$\sum_{x_0}P_{X_{\frac{\pi}{2}}X_0Y_0}(x_{\frac{\pi}{2}},x_0,y_0)=P_{X_{\frac{\pi}{2}}Y_0}(x_{\frac{\pi}{2}},y_0);\quad \sum_{x_{\frac{\pi}{2}}}P_{X_{\frac{\pi}{2}}X_0Y_0}(x_{\frac{\pi}{2}},x_0,y_0)=P_{X_0Y_0}(x_0,y_0).$$