Solid State Theory
Exercise 10

Exercise 10.1 Uniaxial Compressibility

We consider a system of electrons upon which an uniaxial pressure in z-direction acts. Assume that this pressure causes a deformation of the Fermi surface $k \equiv k_F^0$ of the form

$$k_F(\phi,\theta) = k_F^0 + \gamma \frac{1}{k_F^0}\left[3k_z^2 - (k_F^0)^2\right] = k_F^0 + \gamma k_F^0[3\cos^2\theta - 1], \tag{1}$$

where $\gamma = (P_z - P_0)/P_0$ is the anisotropy of the applied pressure.

a) Show that for small $\gamma \ll 1$, the deformed Fermi surface $k_F(\phi,\theta)$ encloses the same volume as the non-deformed one, k_F^0, where terms of order $\mathcal{O}(\gamma^2)$ can be neglected.

b) The deformation of the Fermi surface effects a change in the distribution function of the electrons. Using Landau's Fermi Liquid theory, calculate the uniaxial compressibility

$$\kappa_u = \frac{1}{V}\frac{\partial^2 E}{\partial P_z^2}, \tag{2}$$

which is caused by he deformation given in eq. (1) (E denotes the Landau energy functional).

Exercise 10.2 Polarization of a neutral Fermi liquid

Consider a system of neutral spin-1/2 particles each carrying a magnetic moment $\boldsymbol{\mu} = \frac{\mu}{2}\boldsymbol{\sigma}$. An electric field $\boldsymbol{E}$ couples to the atoms by the relativistic spin-orbit interaction

$$H_{SO} = \frac{\mu}{2}\left(\frac{\boldsymbol{v}}{c} \times \boldsymbol{E}\right)\cdot\boldsymbol{\sigma} \tag{3}$$

where $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ is the vector of Pauli spin matrices. In the following we want to calculate the linear response function χ for the uniform polarization

$$\boldsymbol{P} = \chi\boldsymbol{E}. \tag{4}$$

In the presence of spin-orbit interaction we have to consider a more general situation of a distribution of quasiparticles with variable spin quantization axis. In such a case we must treat the quasiparticle distribution function and the energy as a 2×2 matrix, $(\hat{n}_{\boldsymbol{p}})_{\alpha\beta}$ and $(\hat{\epsilon}_{\boldsymbol{p}})_{\alpha\beta}$, respectively. Furthermore, we require f to be a scalar under spin rotations. In this case f must be of the form

$$\hat{f}_{\alpha\beta,\alpha'\beta'}(\boldsymbol{p},\boldsymbol{p}') = f^s(\boldsymbol{p},\boldsymbol{p}')\delta_{\alpha\beta}\delta_{\alpha'\beta'} + f^a(\boldsymbol{p},\boldsymbol{p}')\boldsymbol{\sigma}_{\alpha\beta}\cdot\boldsymbol{\sigma}_{\alpha'\beta'} \tag{5}$$

a) Expand $\hat{n}_{\boldsymbol{p}}$, $\hat{\epsilon}_{\boldsymbol{p}}$, and $\hat{f}_{\boldsymbol{\sigma}\boldsymbol{\sigma}'}(\boldsymbol{p},\boldsymbol{p}')$ in terms of the unit matrix $\sigma^0 = \mathbf{1}$ and the Pauli spin matrices $\sigma^1 = \sigma^x$, $\sigma^2 = \sigma^y$, $\sigma^3 = \sigma^z$ and find Landau's energy functional E.

b) Assume that the electric field is directed along the z direction. Show that the polarization of such a system is given by

$$P_z = \frac{\partial E}{\partial E_z} = \frac{\mu}{m^* c}\sum_{\boldsymbol{p}}\left(p_y\delta n_{\boldsymbol{p}}^1 - p_x\delta n_{\boldsymbol{p}}^2\right). \tag{6}$$

Here, $\delta n_{\boldsymbol{p}}^i = \frac{1}{2}\mathrm{tr}\,[\delta\hat{n}_{\boldsymbol{p}}\,\sigma^i]$ and $\delta\hat{n}_{\boldsymbol{p}}$ is the deviation from the equilibrium ($E_z = 0$) distribution function.

c) The application of the electric field changes the quasiparticle energy in linear response according to

$$\delta\tilde{\epsilon}^i_{\boldsymbol{p}} = \delta\epsilon^i_{\boldsymbol{p}} + \frac{2}{V}\sum_{\boldsymbol{p}'} f^{ii}(\boldsymbol{p},\boldsymbol{p}')\delta n^i_{\boldsymbol{p}'} \quad \text{with} \quad \delta n^i_{\boldsymbol{p}} = \frac{\partial n_0}{\partial \epsilon}\delta\tilde{\epsilon}^i_{\boldsymbol{p}} = -\delta(\epsilon^0_{\boldsymbol{p}} - \epsilon_F)\delta\tilde{\epsilon}^i_{\boldsymbol{p}}. \tag{7}$$

Use the ansatz $\delta\tilde{\epsilon}^i_{\boldsymbol{p}} = \alpha\delta\epsilon^i_{\boldsymbol{p}}$ and show that $\alpha = 1/(1 + F^a_1/3)$ to find $\delta n^i_{\boldsymbol{p}}$ and $\delta\tilde{\epsilon}^i_{\boldsymbol{p}}$.

d) Compute χ according to Eq. (4).

Office hour:
Monday, May 9th, 2011 - 9:00 to 11:00 am
HIT K 12.1
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