Üb 2.4

The external derivative

We want to show that $d\varphi^* = \varphi^* d$.

a) Let us take the one form $\alpha \in \Omega(N)$ with the mapping $\varphi : M \to N$.

We define the local coordinates of M and N as the following:

$$x = (x^1, \dots, x^m) : M \to \mathbb{R}^m$$

$$y = (y^1, \dots, y^n) : N \to \mathbb{R}^n$$

with $y^j = \varphi^j(x)$.

We can now expand α in local coordinates:

$$\alpha(y) = \sum_{j=1}^{n} \alpha_j(y)\, dy^j$$

where the α_j's are smooth functions ($\alpha_j : N \to \mathbb{R}$).

By definition of the pullback we have

$$(\varphi^*\alpha)(x) = \alpha(\varphi(x))$$

and

$$\begin{cases} \alpha(\varphi(x)) = \sum_{j=1}^{n} \alpha_j(\varphi(x))\, dy^j = \sum_{j=1}^{n} \sum_{i=1}^{m} \alpha_j(\varphi(x)) \frac{\partial \varphi^j(x)}{\partial x^i} dx^i \\ (\varphi^*\alpha)(x) = \sum_{i=1}^{m} (\varphi^*\alpha)_i(x)\, dx^i \end{cases}$$

from which we get

$$(\varphi^*\alpha)_i(x) = \sum_{j=1}^{n} \alpha_j(\varphi(x)) \frac{\partial \varphi^j}{\partial x^i}(x) \qquad (*)$$

b) Let us consider the simple example:

$\omega = df$ where f is the function $f: N \to \mathbb{R}$ with $N \subset \mathbb{R}^3$

and ω the differential form $\omega: TN \to \mathbb{R}$.

Let us be more specific and write df in terms of the local coordinates

$$df = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}\, dx^i$$

Note that the $x^i: N \to \mathbb{R}$ are the coordinate functions (for $i \in \{1,2,3\}$), the $\left\{\frac{\partial}{\partial x^i}\right\}$ constitute a basis for TN and the $\{dx^i\}$ constitute a basis for T^*N.

We define the diffeomorphism $\varphi(y) = x : N \to N$.

We can now compute from (*):

$$(\varphi^* df)(y) = \sum_{i,j=1}^{3} \frac{\partial f}{\partial x^i}(\varphi(y)) \underbrace{\frac{\partial \varphi^i(y)}{\partial y^j} dy^j}_{= dx^i} = df(x)$$

and

$$(d\, \varphi^* f)(y) = d f(\varphi(y)) = d f(x)$$

c) Let us now consider the one form (see a)) $\alpha \in \Omega(N)$ such that

$\omega = d\alpha \in \Omega^2(N)$ with $N \subset \mathbb{R}^3$. In local coordinates

$$\alpha(y) = \sum_{i=1}^{3} \alpha_i(y)\, dy^i = \alpha_1(y)\, dy^1 + \alpha_2(y)\, dy^2 + \alpha_3(y)\, dy^3$$

and

$$\omega(y) = d\alpha(y) = \sum_{i,j=1}^{3} \frac{\partial \alpha_i(y)}{\partial y^j}\, dy^j \wedge dy^i$$

$$= \left(\frac{\partial \alpha_2}{\partial y^1} - \frac{\partial \alpha_1}{\partial y^2}\right) dy^1 \wedge dy^2 + \left(\frac{\partial \alpha_3}{\partial y^2} - \frac{\partial \alpha_2}{\partial y^3}\right) dy^2 \wedge dy^3 + \left(\frac{\partial \alpha_1}{\partial y^3} - \frac{\partial \alpha_3}{\partial y^1}\right) dy^3 \wedge dy^1$$

We can compute now explicitely

$$\left(\varphi^* \omega\right)(x) = \left(\varphi^* d\alpha\right)(x), \qquad \varphi(x) = y$$

$$= \sum_{\substack{i,j=1\\ k,m=1}}^{3} \frac{\partial \alpha_i}{\partial y^j}(\varphi(x)) \frac{\partial \varphi^j(x)}{\partial x^k} \frac{\partial \varphi^i(x)}{\partial x^m} dx^k \wedge dx^m$$

$$= \sum_{i,j=1}^{3} \frac{\partial \alpha_i}{\partial y^j}(y) \; dy^j \wedge dy^i$$

$$= d\alpha\,(y)$$

and

$$(d\varphi^*\alpha)(x) = d\left(\sum_{i=1}^{3} (\varphi^*\alpha)_i(x) \, dx^i\right)$$

$$= d\left(\sum_{i,j=1}^{3} \alpha_j(\varphi(x)) \frac{\partial \varphi^j(x)}{\partial x^i} dx^i\right)$$

$$= \sum_{i,j,k=1}^{3} \frac{\partial}{\partial x^k}\left(\alpha_j(\varphi(x)) \frac{\partial \varphi^j(x)}{\partial x^i}\right) dx^k \wedge dx^i$$

$$= \sum_{\substack{i,j=1\\ k,m=1}}^{3} \frac{\partial \alpha_j(\varphi(x))}{\partial y^m} \frac{\partial \varphi^m(x)}{\partial x^k} \frac{\partial \varphi^j(x)}{\partial x^i} dx^k \wedge dx^i$$

$$+ \underbrace{\sum_{i,j,k=1}^{3} \alpha_j(\varphi(x)) \frac{\partial^2 \varphi^j(x)}{\partial x^k \partial x^i} dx^k \wedge dx^i}$$

$$= \sum_{i,j,k=1}^{3} \frac{1}{2}\left(\alpha_j(\varphi(x)) \frac{\partial^2 \varphi^j(x)}{\partial x^k \partial x^i} dx^k \wedge dx^i + \alpha_j(\varphi(x)) \underbrace{\frac{\partial^2 \varphi^j(x)}{\partial x^i \partial x^k} dx^i \wedge dx^k}_{= -\frac{\partial^2 \varphi^j(x)}{\partial x^k \partial x^i} dx^k \wedge dx^i}\right)$$

and so the last term is 0! We then have

$$(d\,\varphi^* \alpha)\,(x) = \sum_{j,m=1}^{3} \frac{\partial \alpha_j(y)}{\partial y^m}\, dy^m \wedge dy^j$$

$$= d\alpha\,(y) \quad !$$

General case: see next page!

d) Let us now consider $\alpha \in \Omega^{r+1}(N)$ (a(r+1)-form) and $\beta \in \Omega^r(N)$ such that

$$\alpha(y) = (d\beta)(y).$$

We have, by the definition of the exterior derivative,

$$(d\beta)(y) = \sum_{\substack{i_0, i_1, i_2, \\ \dots, i_r}} \frac{\partial \beta_{i_1, \dots, i_r}}{\partial y^{i_0}}(y) \, dy^{i_0} \wedge \dots \wedge dy^{i_r}$$

and now we can compute (from (*))

$$(\varphi^* d\beta)(x) = \sum_{\substack{i_0, i_1, \\ \dots, i_r \\ j_0, \dots, j_r}} \frac{\partial \beta_{i_1, \dots, i_r}}{\partial y^{i_0}}(y = \varphi(x)) \frac{\partial \varphi^{i_0}}{\partial x^{j_0}}(x) \dots \frac{\partial \varphi^{i_r}}{\partial x^{j_r}}(x) \, * \, dx^{j_0} \wedge \dots \wedge dx^{j_r},$$

and

$$(d\, \varphi^* \beta)(x) = \sum_{\substack{i_1, \dots, i_r \\ j_0, \dots, j_r}} \frac{\partial}{\partial x^{j_0}} \left(\beta_{i_1, \dots, i_r}(\varphi(x)) \frac{\partial \varphi^{i_1}}{\partial x^{j_1}}(x) \dots \frac{\partial \varphi^{i_r}}{\partial x^{j_r}}(x) \right) * \, dx^{j_0} \wedge \dots \wedge dx^{j_r}.$$

in which we have the terms

$$\frac{\partial}{\partial x^{j_0}} \beta_{i_1, \dots, i_r}(\varphi(x)) = \sum_{i_0} \frac{\partial \beta_{i_1, \dots, i_r}}{\partial y^{i_0}}(y = \varphi(x)) \frac{\partial \varphi^{i_0}}{\partial x^{j_0}}.$$

in all the other terms we have:

$$\frac{\partial^2 \varphi_{i_k}(x)}{\partial x^{j_0} \partial x^{j_k}} = \frac{\partial^2 \varphi_{i_k}(x)}{\partial x^{j_k} \partial x^{j_0}}$$

but $dx^{j_0} \wedge dx^{j_k} = - dx^{j_k} \wedge dx^{j_0}$ such that they cancel out!