Üb 3.4

Canonical transformations

We want to derive the coordinate transformations $\varphi: M \to M : \{q,p\} \mapsto \{P,Q\}$ knowing that the mapping φ is canonical. ($\varphi(q) = Q$, $\varphi(p) = P$; M is a symplectic manifold with the symplectic structure (closed non-degenerate differential 2-form)

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i \qquad (\dim M = 2n).)$$

We have

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i = \sum_i d\varphi_i(P) \wedge d\varphi^i(Q)$$

$$= \sum_i \varphi^*(dP_i \wedge dQ^i)$$

$$= \varphi^* \tilde{\omega}$$

where $\tilde{\omega} = \sum_{i=1}^{n} dP_i \wedge dQ^i$.

Since φ is canonical the following holds:

$$\varphi^* \tilde{\omega} = \tilde{\omega}$$

from which we get

$$\tilde{\omega} = \sum_i dP_i \wedge dQ^i = \sum_i dp_i \wedge dq^i = \omega .$$

From the definition of the exterior derivative we have:

$$\tilde{\omega} = d\left(\sum_i P_i\, dQ^i\right) = d\left(\sum_i p_i\, dq^i\right) = \omega$$

$$\Leftrightarrow d\left[\sum_i \left(P_i\, dQ^i - p_i\, dq^i\right)\right] = 0 .$$

Making use of the Poincaré-Lemma, we can write:

$$\sum_{i=1}^{n} P_i \, dQ^i - p_i \, dq^i = d\,S(q_1,\dots,q_n,Q_1,\dots,Q_n)$$

$$= \sum_i \frac{\partial S}{\partial Q^i} dQ^i + \frac{\partial S}{\partial q^i} dq^i$$

$$\Rightarrow \quad \frac{\partial S}{\partial Q^i} = P_i \quad ; \quad \frac{\partial S}{\partial q^i} = - p_i \, . \qquad (*)$$

We have found that it exists a function $S = S(q,Q)$ from which the coordinate transformation $\{q,p\} \mapsto \{Q,P\}$ can be derived through the equations $(*)$. Such a function is called a generating function.

The coordinate transformation is well defined if we can inverte the equations $(*)$.

It is the case if $\dfrac{\partial^2 S}{\partial q^i \partial Q^i} \neq 0$, i.e. if

(q,Q) are good local coordinates.

We can choose other pairs of variables (instead of (q,Q)). Let us define the following function:

$$\tilde{S}(P,q) := \sum_i P_i \, \varphi^i(q) = \sum_i P_i \, Q^i ,$$

from which we get

$$d\Big(\sum_i Q^i \, dP_i - \sum_i p_i \, dq^i\Big) = d\tilde{S}(q,P)$$

and

$$\frac{\partial \tilde{S}}{\partial P_i}(q, P) = Q^i \quad ; \quad \frac{\partial \tilde{S}}{\partial q^i}(q, P) = -p_i \ . \quad (**)$$

Note that

$$\frac{\partial \tilde{S}}{\partial P_i} = \varphi^i(q) = Q^i$$

and

$$\frac{\partial \tilde{S}}{\partial q^i} = \sum_j P_j \ \frac{\partial \varphi^j(q)}{\partial q^i} = -p_i \, .$$

We have derived a coordinate transformation through the function $\tilde{S} = \tilde{S}(P, q)$ and the equations $(**)$.

$\tilde{S}$ is again a generating function.