Advanced Topics in Quantum Information Theory
Exercise 1

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Exercise 1.1 The role of initial correlations: beyond CP maps and the Kraus representation

In assessing the dynamics of an open quantum system, the system of interest and its environment are often assumed to be initially in a separable tensor product state. In this case the reduced evolution of the system is known to be described by a completely positive (CP) map. However, as was pointed out by Pechukas[1] and others, this needs not be the case if one takes into account *initial correlations* between the system and its environment. The purpose of this exercise is to convince you with a simple example that such initial correlations play an important role in the reduced dynamics of a quantum system.

Consider two qubit systems interacting via an exchange-type Hamiltonian

$$H = H_0 + H_1$$

with

$$H_0 = \hbar\omega\sigma_{ee}^{(1)} + \hbar\omega\sigma_{ee}^{(2)}$$
$$H_1 = \hbar g(\sigma_{eg}^{(1)}\sigma_{ge}^{(2)} + \sigma_{ge}^{(1)}\sigma_{eg}^{(2)})$$

where we have denoted $\sigma_{ab}^{(i)} \equiv |a\rangle\langle b|_i$ with $a, b \in \{e, g\}$ and $|e\rangle_i$, $|g\rangle_i$ are respectively the ground and the excited states of the i^{th} qubit. In what follows, we shall always refer to the first qubit as the *system* and to the second as the *reservoir*.

a.) Assume that the two qubits are initially in an arbitrary state $\rho(0)$ and find the state $\rho(t)$ at time t. (Hint: To simplify your calculation, solve the Schrödinger equation in the interaction picture with respect to H_0, and transform the state back into the Schrödinger picture in the end).

b.) Express your result in terms of the three following quantities: the initial reduced density matrices $\rho^{(1)}(0)$, $\rho^{(2)}(0)$ of the system (1) and the reservoir (2), and the initial correlations between the two (embedded in the remaining components of the initial density matrix $\rho(0)$ of the complete system (1+2)).

The reduced dynamics of the system after a time t can always be expressed as a linear, homogeneous map

$$\rho_{ab}^{(1)}(t) = \sum_{cd} A_{ab;cd}(t)\rho_{cd}^{(1)}(0)$$

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[1] P. Pechukas, Phys. Rev. Lett. 73, 1060 (1994).

where here A is a 4×4 matrix. One can show[2] that the action of the map A on the initial density matrix can be written as

$$\rho^{(1)}(t) = \sum_i \lambda_i M_i \rho^{(1)}(0) M_i^\dagger$$

with

$$\sum_i \lambda_i M_i^\dagger M_i = I$$

where I denotes the identity matrix, λ_i are the (real) eigenvalues of the matrix A defined above, and M_i are operators obtained from the spectral decomposition of the map A (their explicit form is not important here).
We note that the above expression for the map A is very similar to the Kraus representation of CP maps. In fact, in the case where $\lambda_i \geq 0\ \forall i$, one may define new operators $M_i' = \sqrt{\lambda_i} M_i$ and recover the Kraus representation, thereby showing the map A is a CP map. However, in general some of the λ_i could be *negative*, in which case the map is *not completely positive* (NCP).

c.) Find the eigenvalues λ_i of the map A defined above.

d.) Examine the two cases where the two qubits are: (i) in an initial product state with $\rho^{(2)}(0) = \frac{1}{2}I$ and (ii) in an initial entangled state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|e, g\rangle + e^{i\varphi}|g, e\rangle)$. In both cases, argue on the complete positiveness of the map A.

Exercise 1.2 Error Models in the Lindblad Picture

Under certain conditions the evolution $\rho(t)$ of an open quantum mechanical system can be described by the Lindblad equation

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_i \gamma_i \left(c_i \rho(t) c_i^\dagger - \frac{1}{2} c_i^\dagger c_i \rho(t) - \frac{1}{2} \rho(t) c_i^\dagger c_i \right) , \qquad (1)$$

where H denotes the ordinary system Hamiltonian, the c_i are the Lindblad operators, and the γ_i non-negative constants. In the following we set $H \equiv 0$ and solve (1) for different scenarios.

a.) Consider one qubit and Lindblad operators $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbb{1}_2$ with constants $\gamma_{\sigma_z} = \gamma_{\mathbb{1}_2}$ respectively. This regime corresponds to the error model 'dephasing', where only phase flip errors happen. Solve (1) for this case. What happens for $t \to \infty$? Generalize your results to K qubits, where every single qubit can undergo a phase flip error independently (total dephasing). Is there a decoherence free subspace? Is it possible to store classical information reliably in such a system?

b.) Consider one qubit and Lindblad operators $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbb{1}_2$ with constants $\gamma_{\sigma_x} = \gamma_{\sigma_y} = \gamma_{\sigma_z} = \gamma_{\mathbb{1}_2}$ respectively. This regime

[2] See for example A. R. Usha Devi, A. K. Rajagopal, Sudha, Phys. Rev. A 70, 052110 (2004), sect. II.

corresponds to an error model, where bit flip errors, phase flip errors or both happen. Solve (1) for this case. What happens for $t \to \infty$? Generalize your results to K qubits, where every single qubit can undergo a bit flip error, a phase flip error or both independently. Is there a decoherence free subspace? In the lecture we called this error model total decoherence. Compare your results with the conclusion drawn in the lecture.

c.) Consider two qubits and and Lindblad operators $\sigma_x \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \sigma_x$, $\sigma_y \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \sigma_y$, $\sigma_z \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \sigma_z$, and $\mathbb{1}_2 \otimes \mathbb{1}_2$ with constants $\gamma_{\sigma_x} = \gamma_{\sigma_y} = \gamma_{\sigma_z} = \gamma_{\mathbb{1}_2}$ respectively. This regime corresponds to an error model, where both qubits undergo the same bit flip error, phase flip error or bit and phase flip error combined. Solve (1) for this case. What happens for $t \to \infty$? Is there a decoherence free subspace? This is the simplest case of the collective decoherence. Compare your results with the conclusion drawn in the lecture.

Exercise 1.3 Collective Decoherence

Generalize the collective decoherence model of Exercise 1.2 c.) from two to K qubits and use the results from the lecture to compute the decoherence free subspace when K is even.