Advanced Topics in Quantum Information Theory Solution 4

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Exercise 4.1 Shor code and arbitrary single qubit errors

a.) Let E be parametrized as

$$E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} .$$

By defining the complex numbers e_1, e_2, e_3, e_4 as

$$\begin{aligned} e_1 &:= (E_{11} + E_{22})/2 \\ e_2 &:= (E_{12} + E_{21})/2 \\ e_3 &:= (E_{11} - E_{22})/2 \\ e_4 &:= (E_{21} - E_{12})/2 \end{aligned}$$

yields $E = e_1 \cdot \mathbb{I} + e_2 \cdot X + e_3 \cdot Z + e_4 \cdot X \cdot Z$.

b.) According to the previous item we can write the operator E_j^i as

$$E_j^i = e_1^i \cdot \mathbb{I}_j + e_2^i \cdot X_j + e_3^i \cdot Z_j + e_4^i \cdot X_j \cdot Z_j . \quad (1)$$

As the noise process $\mathcal{N}_j$ is a trace preserving CPM it must hold that $\sum_i (E^i)^\dagger \cdot E^i = \mathbb{I}$, which implies by straightforward calculations that

$$\sum_i \left(|e_1^i|^2 + |e_2^i|^2 + |e_3^i|^2 + |e_4^i|^2\right) = 1 . \quad (2)$$

The error correction phase $\mathcal{C} : \mathcal{S}(\mathcal{H}_C) \to \mathcal{S}(\mathcal{H}_C)$ of the Shor code can be written as

$$\mathcal{C}(\rho) = \sum_s U_s \cdot P_s \cdot \rho \cdot P_s^\dagger \cdot U_s^\dagger ,$$

where $\{P_s\}_s$ denotes a projective measurement with syndrome outcome s and U_s are unitaries which correct the errors depending on the syndrome s. We are allowed to write $\mathcal{C}$ in this form as the syndrome measurements for the bit and phase flips as well as the correction of these errors do all commute (see also Exercise 1.3). Using Stinespring Dilation the trace preserving CPM $\mathcal{C}$ can be represented as an isometry $U_{\mathcal{C}} : \mathcal{H}_C \to \mathcal{H}_C \otimes \mathcal{H}_S$ by

$$U_{\mathcal{C}} := \sum_s U_s \cdot P_s \otimes |s\rangle_S .$$

Let $E_j \in \{\mathbb{I}_j, X_j, Z_j, X_j \cdot Z_j\}$ act on the j'th qubit. Then, because the Shor code protects against the errors X, Z and $X \cdot Z$ on a single qubit it follows that $\mathcal{C}(E_j|\psi\rangle\langle\psi|E_j^\dagger) = |\psi\rangle\langle\psi|$, and therefore

$$U_{\mathcal{C}} E_j |\psi\rangle = |\psi\rangle \otimes |s(E_j)\rangle_S , \quad (3)$$

where $s(E_j)$ denotes the syndrome depending on the error E_j. Let us introduce the following notation $E^i_{j,1} := \mathbb{I}_j$, $E^i_{j,2} := X_j$, $E^i_{j,3} := Z_j$ and $E^i_{j,4} := X_j \cdot Z_j$. Then note that

$$\langle s(E^i_{j,k}) | s(E^i_{j,l}) \rangle = \delta_{k,l} \ , \tag{4}$$

as perfect error correction is possible for the errors X, Z and $X \cdot Z$, i.e., we get different syndromes for different errors.

The noise process $\mathcal{N}_j$ followed by the error correction $\mathcal{C}$ can then be written as

$$\begin{aligned}
\mathcal{C}(\mathcal{N}_j(|\psi\rangle\langle\psi|)) &= \sum_i \sum_{k,l\in\{1,2,3,4\}} e^i_k \cdot (e^i_l)^* \cdot \mathcal{C}(E^i_{j,k}|\psi\rangle\langle\psi|(E^i_{j,l})^\dagger) \\
&= \sum_i \sum_{k,l} e^i_k \cdot (e^i_l)^* \cdot tr_S(U_\mathcal{C} E^i_{j,k}|\psi\rangle\langle\psi|(E^i_{j,l})^\dagger U^\dagger_\mathcal{C}) \\
&= \sum_i \sum_{k,l} e^i_k \cdot (e^i_l)^* \cdot |\psi\rangle\langle\psi| \cdot \langle s(E^i_{j,l})|s(E^i_{j,k})\rangle_S \\
&= \sum_i \sum_k |e^i_k|^2 \cdot |\psi\rangle\langle\psi| \\
&= |\psi\rangle\langle\psi| \ ,
\end{aligned}$$

where we used the linearity of $\mathcal{C}$ in the first line, (3) in the third line, (4) in the fourth line, (2) in the fifth line and that $\mathcal{N}_j(|\psi\rangle\langle\psi|) = \sum_i E^i_j|\psi\rangle\langle\psi|(E^i_j)^\dagger$ with $E^i_j = e^i_1 \cdot E^i_{j,1} + e^i_2 \cdot E^i_{j,2} + e^i_3 \cdot E^i_{j,3} + e^i_4 \cdot E^i_{j,4}$ (see also (1)). Hence, the error correction operation $\mathcal{C}$ of the Shor code can correct errors introduced by arbitrary single qubit CPMs $\mathcal{N}_j$.

Exercise 4.2 Error analysis and concatenation of codes

a.) Proving that

$$\frac{\mathbb{I}}{2} = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4} \ ,$$

immediately implies the statement. So lets do that. Any two-by-two density matrix ρ can be written as

$$\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1| + c|0\rangle\langle 1| + c^*|1\rangle\langle 0| \ ,$$

for some real number $0 \le p \le 1$ and some complex number c. We then have

$$\begin{aligned}
X\rho X &= p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0| + c|1\rangle\langle 0| + c^*|0\rangle\langle 1| \\
Y\rho Y &= p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0| - c|1\rangle\langle 0| - c^*|0\rangle\langle 1| \\
Z\rho Z &= p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1| - c|0\rangle\langle 1| - c^*|1\rangle\langle 0| \ ,
\end{aligned}$$

which implies that $\rho + X\rho X + Y\rho Y + Z\rho Z = 2 \cdot \mathbb{I}$.

b.) Let us first compute a lower bound on the probability that an error occurs which can be corrected. Of course, if no error occurs we are fine. This happens with probability

$$(1-p)^9 \ ,$$

as the noise is acting independently on each qubit. A single X, Z or Y flip can also be corrected by the Shor code. This happens with probability

$$9 \cdot \frac{p}{3}(1-p)^8 + 9 \cdot \frac{p}{3}(1-p)^8 + 9 \cdot \frac{p}{3}(1-p)^8 \ .$$

Exactly two bit flips can be corrected in 27 (the bit flips have to be in different blocks) out of the total $36 = (9 \cdot 8)/2$ cases. Hence, the probability that these errors can be corrected is

$$27 \cdot \left(\frac{p}{3}\right)^2 (1-p)^7 \ .$$

Exactly two phase flips can be corrected in only 9 (the phase flips have to be in the same block) out of the total 36 cases. Hence, the probability that these errors can be corrected is

$$9 \cdot \left(\frac{p}{3}\right)^2 (1-p)^7 \ .$$

Exactly two Y flips cannot be corrected by the Shor code. Note that even if there are more than two errors it is still possible that the Shor code protects against these errors. By adding up the above probabilities gives us therefore a lower bound. Hence, the probability that the error can be corrected is larger than

$$(1-p)^9 + 9 \cdot p \cdot (1-p)^8 + 4 \cdot p^2 \cdot (1-p)^7 \approx 1 - 32 \cdot p^2 \ ,$$

where we neglected higher order terms, and therefore the probability that an error occurs which cannot be corrected is

$$\lesssim 32 \cdot p^2 \ .$$

c.) Note first that we cannot use the analysis we did in the previous item to solve this one. The reason is that although we have the error process $\mathcal{N}$ at the first concatenation level this does not imply that this same error process is acting on the second concatenation level as well (where each of the nine qubits in the Shor code is represented itself by nine qubits of the Shor code, i.e., there is a total of 81 qubits at the second level). At each concatenation level we have a different error process $\mathcal{N}^i$ acting on the (logical) qubits given by

$$\mathcal{N}^i(\rho) := (1-p_i) \cdot \rho + p_i \cdot \tilde{\mathcal{N}}^i(\rho) \ ,$$

with $\mathcal{N}^1(\rho) := \mathcal{N}(\rho) = (1-p)\rho + p/3(X\rho X + Y\rho Y + Z\rho Z)$. Note, however, that the error process is still acting independently on the (logical) qubits and therefore the overall noise process at the i'th concatenation level is described by $(\mathcal{N}^i)^{\otimes 9}$.

The goal is now to determine a lower bound on the probability that an error occurred which can be corrected at the i'th concatenation level where the noise process $(\mathcal{N}^i)^{\otimes 9}$ is acting on the nine logical qubits. First, the probability that no error occurs is

$$(1-p_i)^9 \ .$$

We know that the Shor code can correct arbitrary single qubit errors. Hence, the probability that exactly a single error occurs, which can then be corrected, is

$$9 \cdot (1-p_i)^8 \cdot p_i \ .$$

Therefore, the probability that the error can be corrected is larger than

$$(1 - p_i)^9 + 9 \cdot (1 - p_i)^8 \cdot p_i \approx 1 - 36 \cdot p_i^2 \ ,$$

where we neglected higher order terms in the calculations, and hence, the error probability is given by

$$p_{i+1} \lesssim 36 \cdot p_i^2 \ .$$

Solving this recursive formula yields for the error probability, given n concatenation levels, the following upper bound

$$p_n \leq \frac{1}{36} \cdot (36 \cdot p)^{2^n} \ .$$

If we set $p < 1/36$ this expression goes to zero by increasing the number of concatenation levels n.