Advanced Topics in Quantum Information Theory
Solution 7

FS 12
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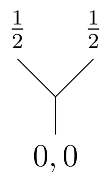
Exercise 7.1 Schur–Weyl Duality

a) The Clebsch–Gordan series for the tensor product of two spin-half representations is just

$$\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathrm{Sym}^2(\mathbb{C}^2) \oplus \mathrm{Alt}^2(\mathbb{C}^2)$$

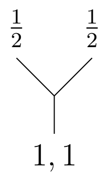
and we will write $|0\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$ and $|1\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$ for the weight vectors of the spin-$\frac{1}{2}$ representation of SU(2).

The trivial representation in the tensor product of two spin-half representations is just the singlet representation $\mathrm{Alt}^2(\mathbb{C}^2) \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2$, so that

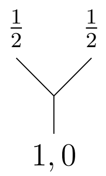


$$= \frac{1}{\sqrt{2}} \left(|01\rangle - |10\rangle\right).$$

The other component is the triplet representation, $\mathrm{Sym}^2(\mathbb{C}^2) \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2$. Therefore,



$$= |00\rangle \quad \Rightarrow$$

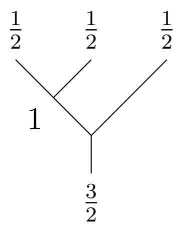


$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot |00\rangle = \frac{1}{\sqrt{2}} \left(|10\rangle + |01\rangle\right).$$

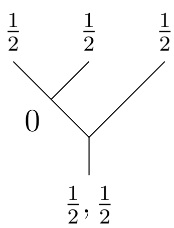
For a triple tensor product of spin-$\frac{1}{2}$ representations, the Clebsch–Gordan series is given by

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cong V_{3/2} \oplus V_{1/2} \oplus V_{1/2}.$$

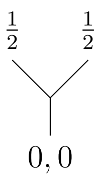
The spin-$\frac{3}{2}$ representation is of course given by $\mathrm{Sym}^3(\mathbb{C}^2) \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, corresponding to the morphism



For the spin-$\frac{1}{2}$ representation, there are two choices: Using the left-handed fusion tree basis, we clearly have

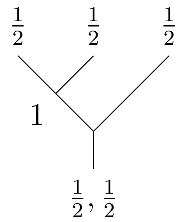
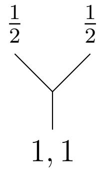
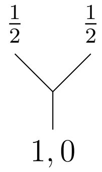


$$=$$

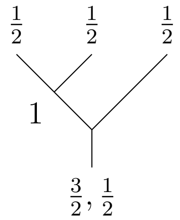


$$\otimes |0\rangle = \frac{1}{\sqrt{2}} \left(|100\rangle - |010\rangle\right),$$

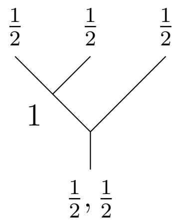
while

 $= \alpha$  $\otimes |1\rangle + \beta$  $\otimes |0\rangle = \alpha\,|001\rangle + \dfrac{\beta}{\sqrt{2}}\left(|100\rangle + |010\rangle\right)$

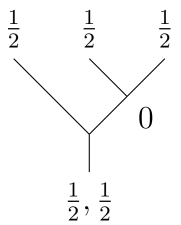
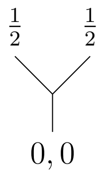
for some constants $\alpha > 0$ (cf. our conventions!) and β satisfying $|\alpha|^2 + |\beta|^2 = 1$. They can be determined from the fact that the vector has to be orthogonal to $\mathrm{Sym}^3(\mathbb{C}^2)$, in particular to

 $= \dfrac{1}{\sqrt{3}}\left(|001\rangle + |100\rangle + |010\rangle\right).$

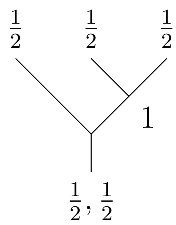
It follows that

 $= \dfrac{1}{\sqrt{6}}\left(2\,|001\rangle - |100\rangle - |010\rangle\right).$

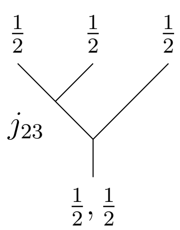
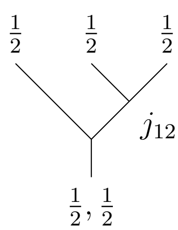
Similarly, one computes using the right-handed fusion tree basis that

 $= |0\rangle \otimes$  $= \dfrac{1}{\sqrt{2}}\left(|010\rangle - |001\rangle\right)$

and

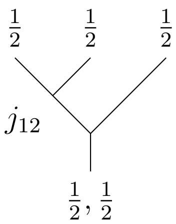
 $= \dfrac{1}{\sqrt{6}}\left(|010\rangle + |001\rangle - 2\,|100\rangle\right).$

b) It follows directly from a) that

$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & j_{12} \\ \frac{1}{2} & \frac{1}{2} & j_{23} \end{bmatrix} = \Big\langle$  $\Big|$  $\Big\rangle = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}_{j_{23}, j_{12}}.$

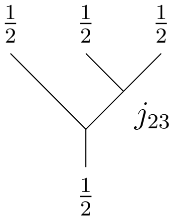
We observe that the right-hand side matrix is a unitary matrix (as expected, since it implements a change of orthonormal bases).

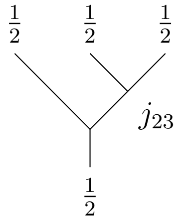
c) Let us write

$$|\overline{j_{12}}\rangle =$$ 

.

We have

$$(2\ 3) \cdot |\overline{j_{12}}\rangle = \sum_{j_{23}} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & j_{12} \\ \frac{1}{2} & \frac{1}{2} & j_{23} \end{bmatrix} (2\ 3) \cdot$$ 

$$= \sum_{j_{23}} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & j_{12} \\ \frac{1}{2} & \frac{1}{2} & j_{23} \end{bmatrix} (-1)^{\frac{1}{2}+\frac{1}{2}-j_{23}}$$ 

$$= \sum_{j_{23}, \tilde{j}_{12}} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & j_{12} \\ \frac{1}{2} & \frac{1}{2} & j_{23} \end{bmatrix} (-1)^{\frac{1}{2}+\frac{1}{2}-j_{23}} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \tilde{j}_{12} \\ \frac{1}{2} & \frac{1}{2} & j_{23} \end{bmatrix}^* |\overline{\tilde{j}_{12}}\rangle .$$

In other words, the action of (2 3) on the basis vectors $|j_{12}\rangle$ is given by the unitary matrix

$$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} .$$

Moreover, (1 2) acts of course by multiplication with $(-1)^{\frac{1}{2}+\frac{1}{2}-j_{12}}$.

It is now easy to see that the isomorphism $\mathcal{H}_{1/2} \to \mathcal{K}$ defined by

$$|\overline{0}\rangle \mapsto \frac{1}{\sqrt{2}}(1, -1, 0)$$
$$|\overline{1}\rangle \mapsto \frac{1}{\sqrt{6}}(-1, -1, 2)$$

is S_3-linear, since the target vectors transform in the correct way under the action of the generators (1 2) and (1 3).

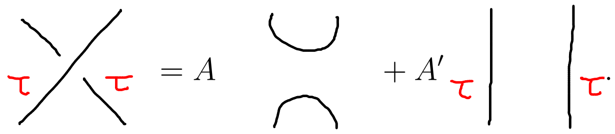
The same is true for the corresponding basis vectors of $\mathcal{H}_{-1/2}$, since the action of SU(2) commutes with the action of S_3.

Exercise 7.2 Fibonacci Anyons

a) This is almost by definition since the braid group acts on the Fibonacci model.

b) This follows directly from (2), i.e., from the fact that the quantum dimension of $\boldsymbol{\tau}$ is equal to ϕ.

c) Since $\boldsymbol{\tau} \otimes \boldsymbol{\tau} = \mathbf{1} \oplus \boldsymbol{\tau}$, the space of diagrams $\boldsymbol{\tau} \otimes \boldsymbol{\tau} \to \boldsymbol{\tau} \otimes \boldsymbol{\tau}$ is two-dimensional. Therefore, there exist constants A and A' such that



By fusing the two $\boldsymbol{\tau}$ anyons and using the braiding phases and the quantum dimensions for the Fibonacci model, we find that

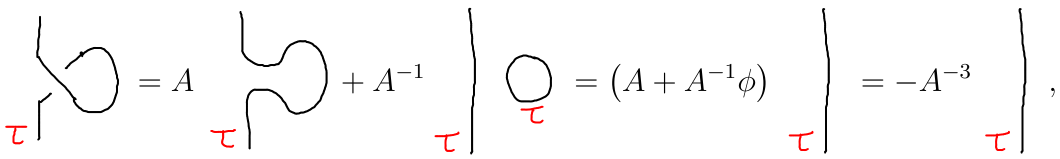
$$A' = R^{\boldsymbol{\tau},\boldsymbol{\tau}}_{\boldsymbol{\tau}} = e^{-3\pi i/5}$$

$$A\phi + A' = R^{\boldsymbol{\tau},\boldsymbol{\tau}}_{\mathbf{1}} = e^{4\pi i/5}$$

Indeed, it follows that

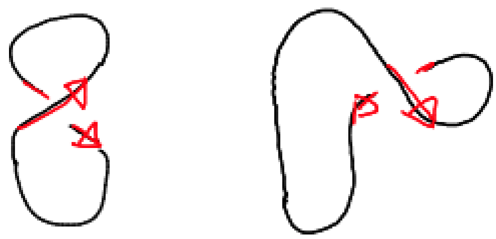
$$A = \frac{e^{4\pi i/5} - A'}{\phi} = e^{3\pi i/5} = (A')^{-1}.$$

d) Clearly,



where we have used that $\phi = -A^2 - A^{-2}$. The analog is true for the other Reidemeister move of type I.

e) The problem is that we would like to determine from the original link diagram $\overline{B}$ the factor by which one has to correct the bracket in order to make it a knot invariant. Naively, one might think that we should simply raise $(-A^3)$ to the number of overcrossings minus the number of undercrossings. However, this number depends on the concrete presentation of the link! E.g., consider the following two diagrams which corresponds to the same knot:



The left-hand side overcrossing got transformed into the right-hand side undercrossing! (Note that this transformation is *not* a Reidemeister move; it is simply a smooth deformation of the link diagram which does not involve manipulating any crossing.)

One way out is to consider links equipped with an orientation (indicated in the picture by the little red arrows). This allows us to see that in fact both crossings in the above picture are to be considered to be of the same type! Formalizing this idea leads to the definition of the writhe.

f) By using e) and the procedure outlined in c) one can compute that

$$\begin{aligned} &V_{\tau_R}(t) = t + t^3 - t^4 \\ \neq\ &V_{\tau_L}(t) = t^{-1} + t^{-3} - t^{-4}, \end{aligned}$$

for $t = e^{-2\pi i/5}$.

Remark: Observe that one cannot distinguish a trefoil from its mirror image by evaluating at $t = e^{2\pi i/r}$ with $r = 1, 2, 3, 4$.