Symmetries, Rotations & Spin. ①

Objects in QM describing some "physical reality"

state $|\psi\rangle$
observable A

Fundamental Question: How can we describe symmetries in QM?

for ex: how do you "rotate by 90°" a state / an observable?

Such a rule $U(R)$ is called a representation of a symmetry group.
(This is where the physics comes in!!)

Properties.

* $U(R)$ is unitary (to preserve probabilities)
* $U(R_1 R_2) = U(R_1)\,U(R_2)$ } compatibility with
* $U(R^{-1}) = U(R)^\dagger$ } the group structure

if these conditions hold up to a phase, then $U(R)$ is a projective representation.

Example. 1 free particle, wave function $\psi(x)$, observables $\hat{X}$ & $\hat{P}$

The rotation R has to act as $\hat{R}\psi(\vec{x}) = \psi(R^{-1}\vec{x})$
Look at small angles $R(d\alpha)$ eg. around z axis.

$$\psi(R^{-1}\vec{x}) = \psi(R_z(-d\alpha)\,\vec{x}) \approx \psi(x + d\alpha\, y,\ y - d\alpha\, x,\ z)$$ ∂ Taylor 1st order

$$\approx \psi(x,y,z) + \frac{\partial\psi}{\partial x}\cdot d\alpha\, y - \frac{\partial\psi}{\partial y}\cdot d\alpha\, x$$

$$= \left[1 + d\alpha\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\right]\psi(\vec{x})$$

$$\longrightarrow \quad \hat{R} \approx 1 + d\alpha\underbrace{\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)}_{= iL_z\ !}$$

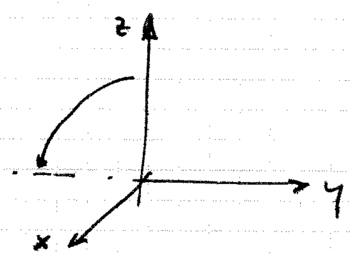
Example 2. Spin-1/2 particle, $\mathcal{H} = \mathbb{C}^2$, $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$

Observables: $\sigma_x, \sigma_y, \sigma_z$ (Pauli Matrices) $\quad [\sigma_x, \sigma_y] = i\sigma_z \quad 2$

$\hookrightarrow$ eigenstates $|\uparrow\rangle$ & $|\downarrow\rangle$

We want $\vec{\sigma}$ to rotate like a vector. (i.e. $\sigma_x, \sigma_y, \sigma_z$ "rotate into each other")

for ex. $\hat{R}_x(\frac{\pi}{2})|\uparrow\rangle = |-y\rangle$



Rotate an observable?

$|\psi\rangle \to \hat{R}|\psi\rangle$.

Conserve expectation values (physically measureable)

$$\langle\psi|A|\psi\rangle \overset{!}{=} \langle\psi'|A'|\psi'\rangle = \langle\psi|\underbrace{\hat{R}^\dagger A'\hat{R}}|\psi\rangle \longrightarrow A = \hat{R}^\dagger A' \hat{R}$$

$\rightarrow \underline{A' = \hat{R}A\hat{R}^\dagger}$ transformation of observables.

Here: $\sigma_z \longrightarrow \hat{R}\sigma_z\hat{R}^\dagger \underset{\text{requ.}}{\overset{!}{=}} (R^{-1}\vec{e}_z)\cdot\vec{\sigma}$ $\quad$ eg. $R = R_x(d\alpha)$

$$(R^{-1}\vec{e}_z)\cdot\vec{\sigma} \approx (\vec{e}_z - d\alpha\,\vec{e}_y)\cdot\vec{\sigma} = \sigma_z - d\alpha\,\sigma_y = \sigma_z + i d\alpha[\sigma_x, \sigma_z]$$

$$\approx \underbrace{(1 + i d\alpha\,\sigma_x)}_{\hat{R}}\,\sigma_z\,\underbrace{(1 - i d\alpha\,\sigma_x)}_{\hat{R}^\dagger}$$

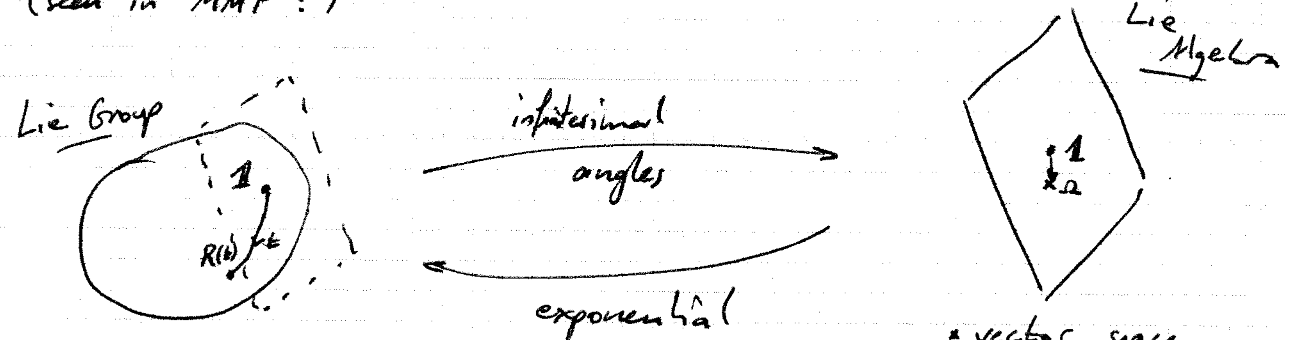
$\longrightarrow \hat{R} \approx 1 + i d\alpha\,\underbrace{\sigma_x}_{\text{angular momentum operator again!}}$

<u>Message</u>. $\boxed{\text{Spin Operators generate the rotations.}}$

This comes from Lie group/algebra theory.

Lie Groups & Algebras - (Matrix groups)

(seen in MMP ?)



Group "parametrized by continuous variables" (think matrix group)

× vector space
× commutator $[\cdot,\cdot]$

Group → Alg. $\quad R(t) \xrightarrow[\text{angles}]{\text{small}} R(dt) \approx 1 + \underbrace{\frac{dR}{dt}\Big|_{t=0}}_{\text{generator } \Omega} \cdot dt$

Generator Ω ($\in$ Lie Algebra) gives the "direction in which $R(dt)$ deviates from $\mathbb{1}$".

$\Omega = \frac{dR}{dt}\Big|_{t=0}$ $\qquad$ Physicists' convention: $M = -i \frac{dR}{dt}\Big|_{t=0}$

hermitian.

Alg. → Group. $\quad \Omega \xrightarrow{\exp} e^{\Omega t} = 1 + \Omega t + \ldots \quad (= \ldots \quad) = R(t)$

[not always!] not obvious.

[idea of the proof: decompose $e^{\Omega t}$ into infinitesimal bits.]

Example. Group $= SO(2) = \{\text{rotations in the } \mathbb{R}^2 \text{ plane}\} = \left\{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\right\}$

Lie alg.? infinitesimal φ →

$$R(d\varphi) = \begin{pmatrix} \cos d\varphi & -\sin d\varphi \\ \sin d\varphi & \cos d\varphi \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\text{Generator } = \Omega} d\varphi$$

Back to Lie group?

$$\exp\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \varphi\right] = ? \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\varphi\right)^n = \underbrace{\sum_{n \text{ even}} \frac{(-1)^{n/2}\varphi^n}{n!}}_{=\cos\varphi \,!!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \underbrace{\sum_{n \text{ odd}} \frac{(-1)^{\frac{n-1}{2}}\varphi^n}{n!}}_{=\sin\varphi \,!!} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

<u>Remark</u>. Role of the commutator as acting on observables.

Say you transform $A \longrightarrow RAR^\dagger$ (<u>conjugation</u>)

Look at infinitesimal angles $\rightarrow$

$$RAR^\dagger \approx (1 - iM\,d\alpha)\,A\,(1 + iM\,d\alpha)$$
$$\approx A - i\,d\alpha \underbrace{[M, A]}_{\text{action by commutator}}$$

$\rightsquigarrow$ $[M, A]$ tells you the infinitesimal change of $A \rightarrow RAR^\dagger$.