Bell Inequalities

- I assume familiarity with basic Quantum Information (i.e. Nielsen/Chuang)

How strong are quantum correlations?

- Bell inequalities
  - as games
  - geometrically

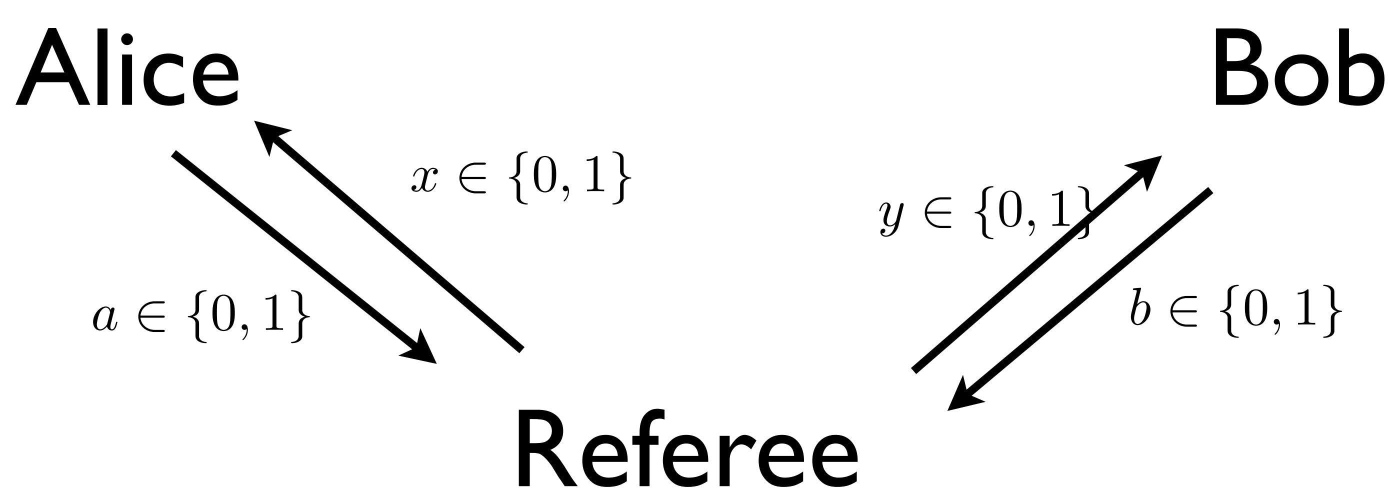
CHSH-Game

Clauser, Horne, Shimony & Holt

Alice and Bob cannot communicate

referee supplies questions to Alice and Bob (x and y)

(equal probability for all questions)



they win if
$a \oplus b \neq x.y$

Alice and Bob win if

a≠b for x=0, y=0
a≠b for x=0, y=1
a≠b for x=1, y=0
a=b for x=1, y=1

What is the maximal probability of winning?

Mathematical setup

- Alice and Bob have access to correlations given by $P_{AB|XY}(ab|xy)$ ← conditional probabilty distribution
- Winning condition $1 \geq Q_{ABXY}(abxy) \geq 0$
- Questions are chosen with probability $P_{XY}(xy)$

$$\text{Prob}[\text{win}] = \sum_{xyab} P_{XY}(xy) P_{AB|XY}(ab|xy) Q(abxy)$$

Classical Deterministic Strategy

$$P_{AB|XY}(ab|xy) = \delta_{a,f(x)}\delta_{b,g(y)}$$

Alice   a=f(x)

Bob   b=g(y)

Example strategy   f(0)=0 f(1)=0 g(0)=1 g(1)=1

Alice and Bob win if

a≠b for x=0, y=0	100%
a≠b for x=0, y=1	100%
a≠b for x=1, y=0	100%
a=b for x=1, y=1	0%

Average winning probability 75%

Optimality

- There is no better strategy
- At most three conditions are satisfied

$$g(0) \neq f(1) = g(1) \neq f(0)$$

3rd 4th 2nd

is in conflict with 1st: $g(0) \neq f(0)$

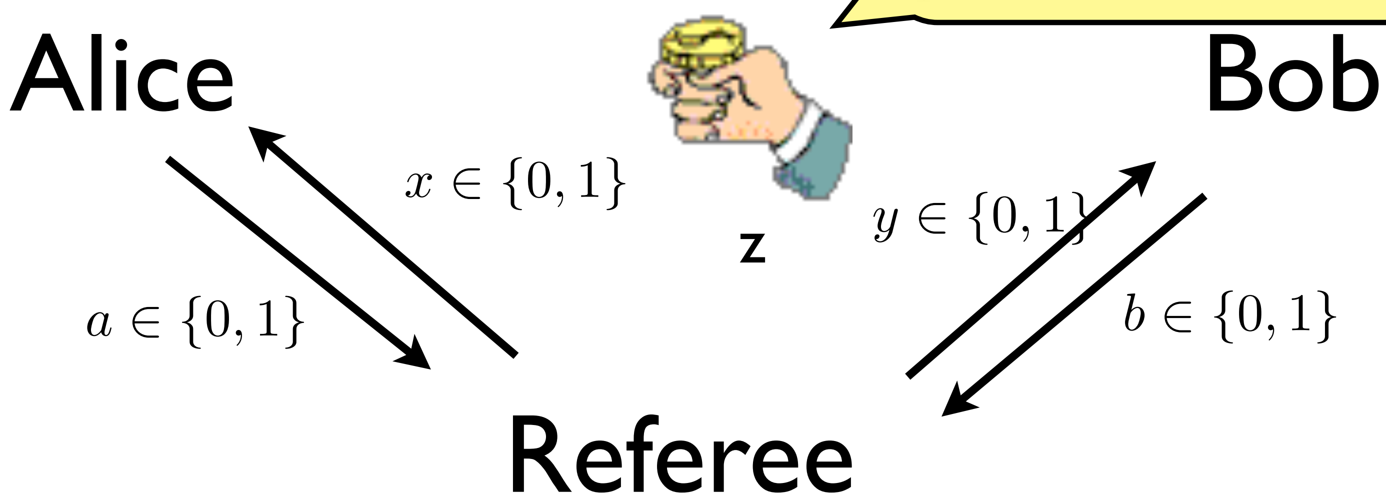
- Winning probability $\mathrm{Prob}[\mathrm{win}|\mathrm{det}] \leq 3/4$

Shared Randomness

deterministic strategies

$$P_{AB|XY}(ab|xy) = \sum_z P_Z(z) P_{A|XZ}(a|xz) P_{B|YZ}(b|yz)$$

shared randomness (local hidden variable)



Alice and Bob win if

a≠b for x=0, y=0
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Bell inequality

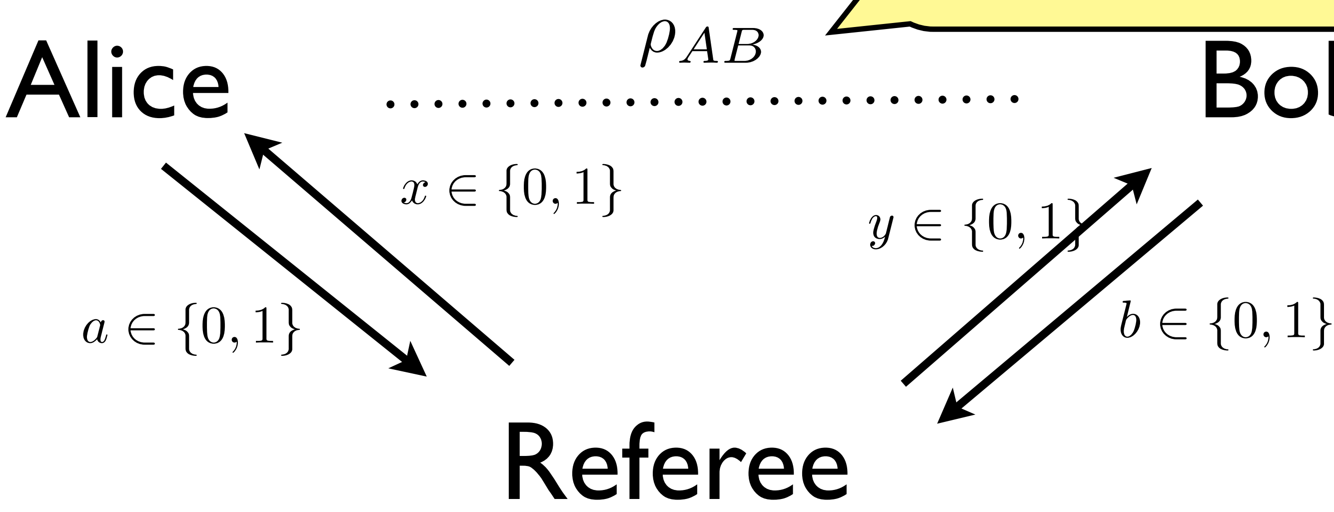
$$\text{Prob}[\text{win}|\text{shared randomness}] \leq 3/4$$

Convex combination of classical deterministic strategies -> fixed deterministic strategy is best

Can we beat this with shared entanglement?

Bell inequality

Quantum Strategy



shared quantum state

POVM

$A_a^x \geq 0 \quad \sum_a A_a^x = \mathbf{1}$

$$P_{AB|XY}(ab|xy) = \mathrm{tr}\ \rho_{AB} A_a^x \otimes B_b^y$$
$$= \mathrm{tr}\ |\psi\rangle\langle\psi| A_a^x \otimes B_b^y$$

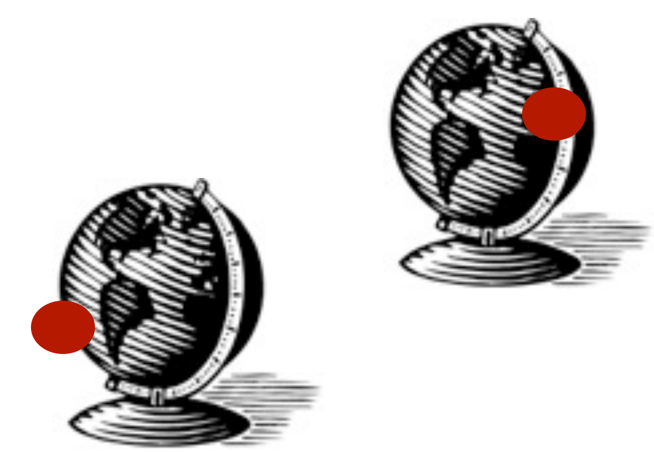
best to choose pure state
(cf. convex combination)

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \qquad |\psi\rangle\langle\psi| = \frac{1}{4}(\mathbf{1}\otimes\mathbf{1} - \sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y - \sigma_z\otimes\sigma_z)$$

$$\mathrm{tr}_A A_a^x \otimes \mathbf{1}|\psi\rangle\langle\psi| = \frac{1}{2}\cdot\rho_{B,x,a}$$

measurement
projector

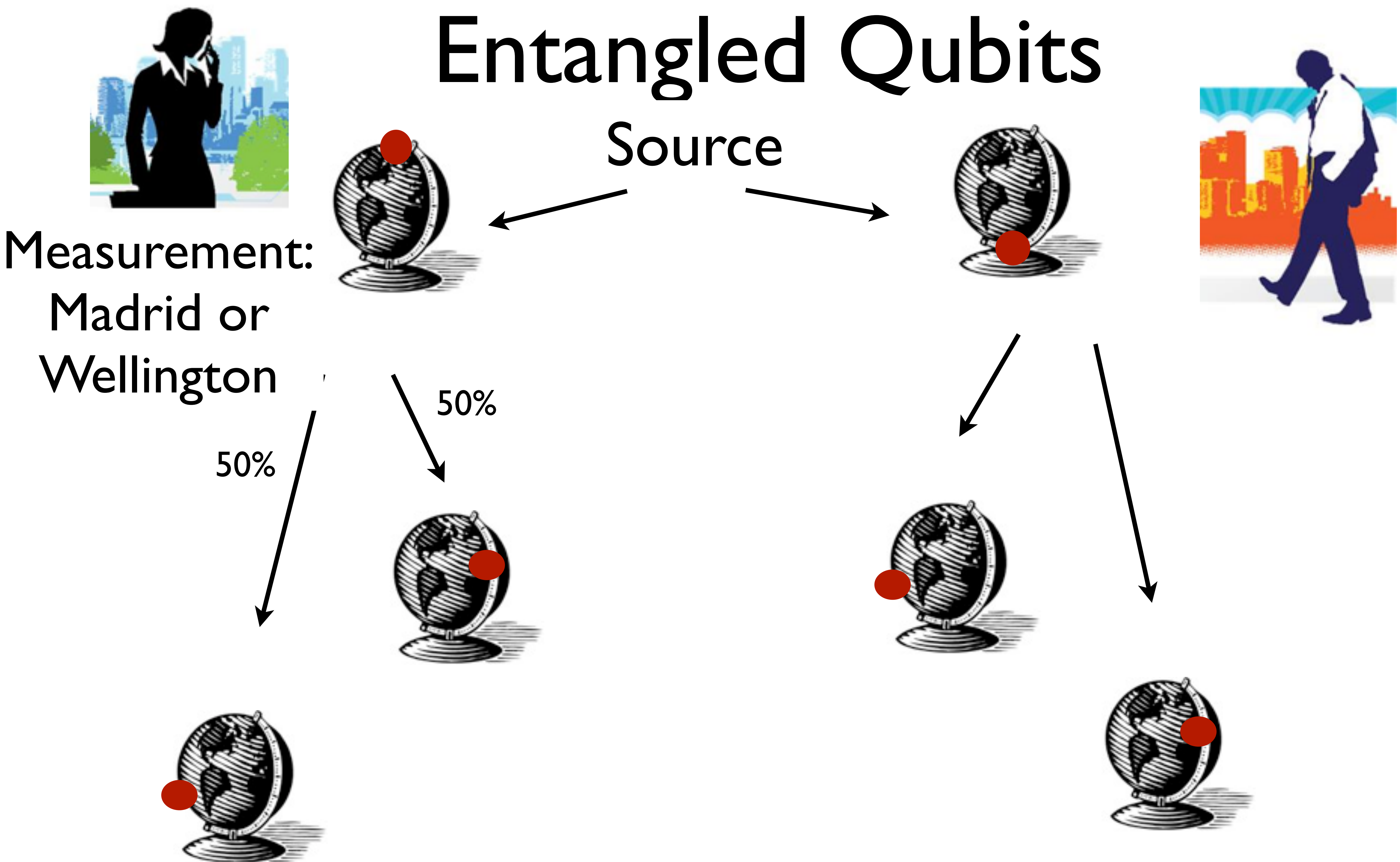
probability of
result



$$= \mathrm{tr}_A \frac{1}{2}(\mathbf{1} + \vec{r}\cdot\vec{\sigma})|\psi\rangle\langle\psi| = \frac{1}{2}\left(\frac{1}{2}(\mathbf{1} - \vec{r}\cdot\vec{\sigma})\right)$$

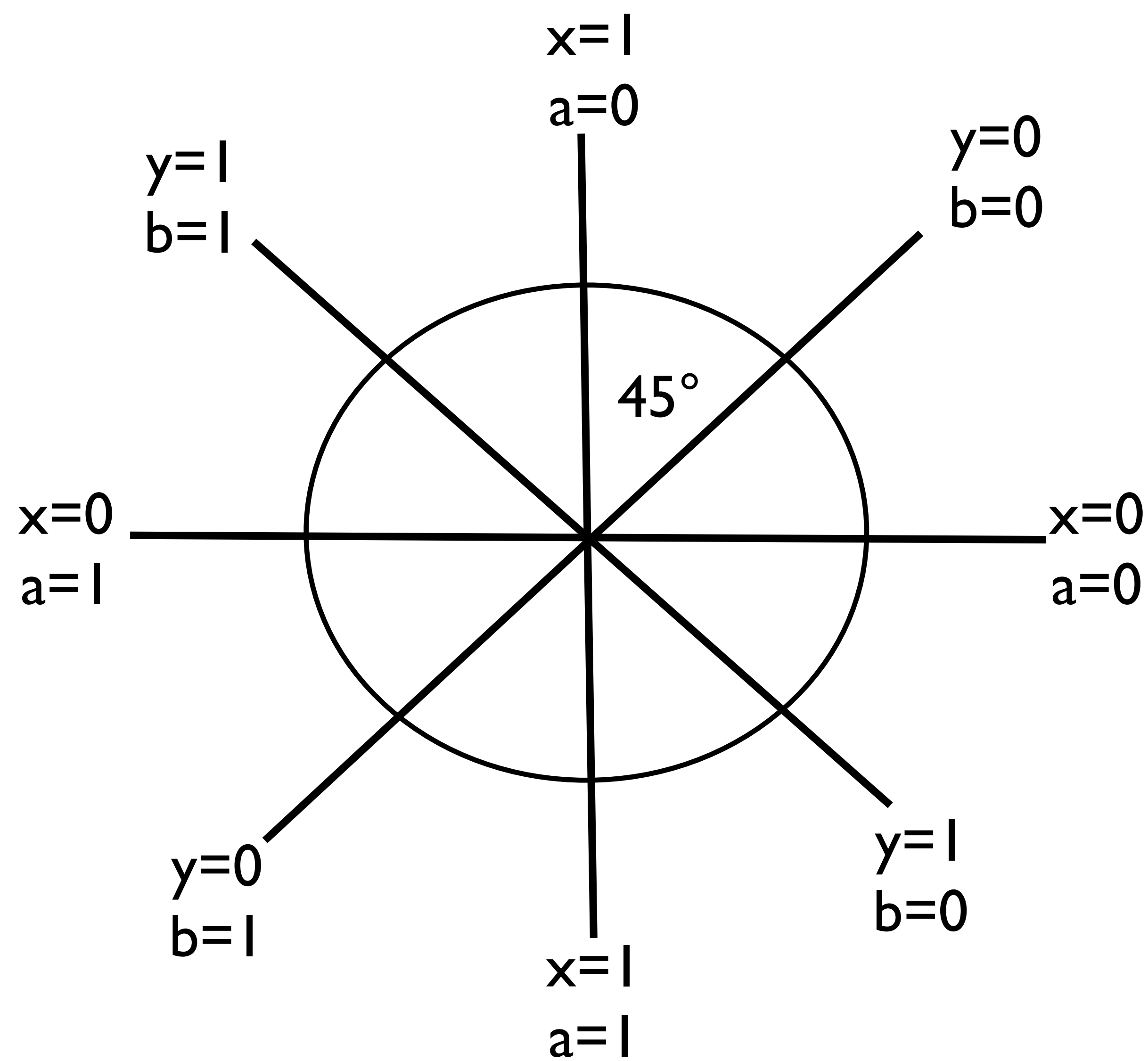
post-measurement state is
antipodal point

Entangled Qubits



Bob‘s state=antipodal point
for every measurement
„spooky action at a distance“

Violating the CHSH inequality



multiple of 45°

$$\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$$

a≠b for x=0, y=0 $\cos^2 45°/2 = \frac{1}{2}(1+\frac{1}{\sqrt{2}}) \approx 85\%$

a≠b for x=0, y=1 $\cos^2 45°/2 \approx 85\%$

a≠b for x=1, y=0 $\cos^2 45°/2 \approx 85\%$

a=b for x=1, y=1 $1-\cos^2 135°/2 \approx 85\%$

$$\text{Prob}[\text{win}] = \frac{1}{2}(1+\frac{1}{\sqrt{2}})$$

Cirelson‘s bound

$$\text{Prob}[\text{win}|\text{quantum}] \leq \frac{1}{2}(1 + \frac{1}{\sqrt{2}})$$

Define $A^x = A_0^x - A_1^x$

$B^x = B_1^y - B_0^y$

orthogonal projectors

$$\text{Prob}[\text{win}] = 1/2 + \langle\psi|A^0 \otimes B^0 + A^0 \otimes B^1 + A^1 \otimes B^0 - A^1 \otimes B^1|\psi\rangle/8$$

$$=1/8 \sum_{abxy} P_{AB|XY}(ab|xy)$$

$$+ (P_{AB|XY}(01|00) + P_{AB|XY}(10|00)) - P_{AB|XY}(00|00) - P_{AB|XY}(11|00))$$

$$+ (P_{AB|XY}(01|01) + P_{AB|XY}(10|01)) - P_{AB|XY}(00|01) - P_{AB|XY}(11|01))$$

$$+ (P_{AB|XY}(01|10) + P_{AB|XY}(10|10)) - P_{AB|XY}(00|10) - P_{AB|XY}(11|10))$$

$$+ (P_{AB|XY}(00|11) + P_{AB|XY}(11|11)) - P_{AB|XY}(01|11) - P_{AB|XY}(10|11))$$

Cirelson‘s bound

Define $a^x = A^x \otimes \mathbf{1}$ Note $(a^x)^2 = \mathbf{1} = (b^y)^2$

$b^y = \mathbf{1} \otimes B^y$ $[a^x, b^y] = 0$

Lemma: Let $(a^x)^2 = \mathbf{1} = (b^y)^2 \quad [a^x, b^y] = 0$

Then $a^0b^0 + a^0b^1 + a^1b^0 - a^1b^1 \leq 2\sqrt{2}\mathbf{1}$

Proof:

$$
\begin{aligned}
a^0b^0 + a^0b^1 + a^1b^0 - a^1b^1 &= \frac{1}{\sqrt{2}}\Big((a^0)^2 + (a^1)^2 + (b^0)^2 + (b^1)^2\Big) \\
&- \frac{\sqrt{2}-1}{8}\Big[\Big((\sqrt{2}+1)(a^0 - b^0) + a^1 - b^1\Big)^2 \\
&\quad + \Big((\sqrt{2}+1)(a^0 - b^1) - a^1 - b^0\Big)^2 \\
&\quad + \Big((\sqrt{2}+1)(a^1 - b^0) + a^0 + b^1\Big)^2 \\
&\quad + \Big((\sqrt{2}+1)(a^1 + b^1) - a^0 - b^0\Big)^2\Big] \\
&\leq \frac{1}{\sqrt{2}}\Big((a^0)^2 + (a^1)^2 + (b^0)^2 + (b^1)^2\Big) \\
&\leq 2\sqrt{2}\mathbf{1}
\end{aligned}
$$

verify with Mathematica

squares are non-negative

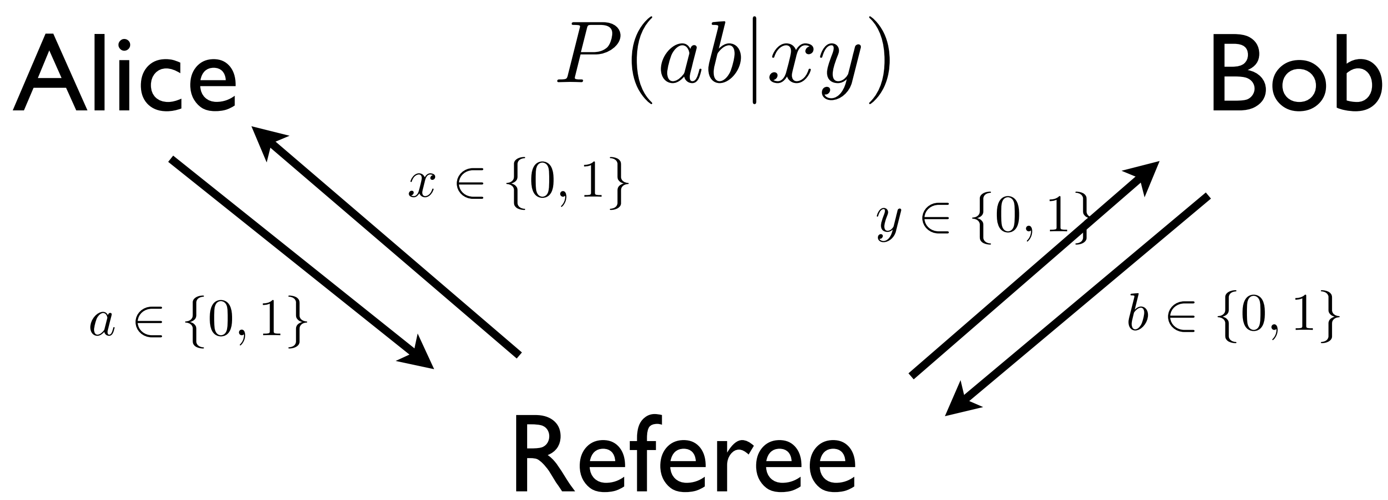
qed

Cirelson‘s bound

Characterisation of winning probability
& Lemma gives Cirelson‘s bound:

$$\begin{aligned}\text{Prob}[\text{win}] &= 1/2 + \langle\psi|A^0 \otimes B^0 + A^0 \otimes B^1 + A^1 \otimes B^0 - A^1 \otimes B^1|\psi\rangle/8 \\ &\leq \frac{1}{2} + \frac{2\sqrt{2}\langle\psi|\mathbf{1}|\psi\rangle}{8} = \frac{1}{2}(1 + \frac{1}{\sqrt{2}})\end{aligned}$$

Non-signaling distributions



Only requirement on $P(ab|xy)$ is that Alice and Bob cannot communicate (no signaling condition)

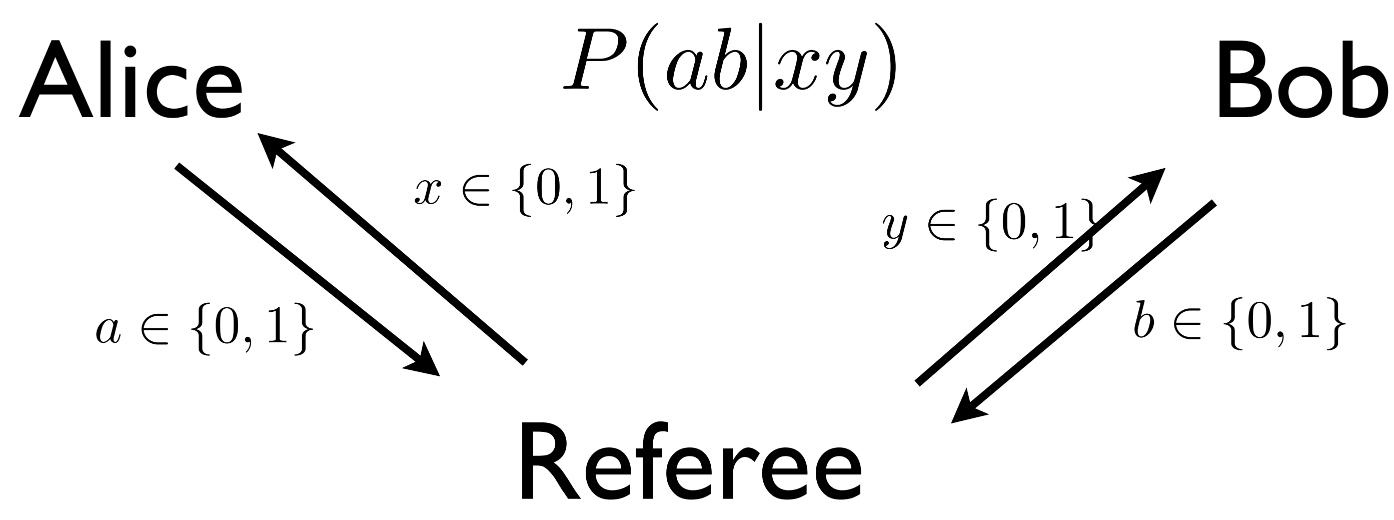
Description of Bob's system alone independent of Alice's measurement

reduced state

$$P_B(b|y) = \sum_a P(ab|0y) = \sum_a P(ab|1y)$$

$$P_A(a|x) = \sum_b P(ab|x0) = \sum_b P(ab|x1)$$

Popescu-Rohrlich (PR) Box



$$P(01|00) = P(10|00) = \frac{1}{2}$$
$$P(01|01) = P(10|01) = \frac{1}{2}$$
$$P(01|10) = P(10|10) = \frac{1}{2}$$
$$P(00|11) = P(11|11) = \frac{1}{2}$$

state is non-signaling:

$$P_B(b|y) = \sum_a P(ab|0y) = \sum_a P(ab|1y) \quad = \frac{1}{2}$$
$$P_A(a|x) = \sum_b P(ab|x0) = \sum_b P(ab|x1) \quad = \frac{1}{2}$$

Alice and Bob win if

a≠b for x=0, y=0
a≠b for x=0, y=1
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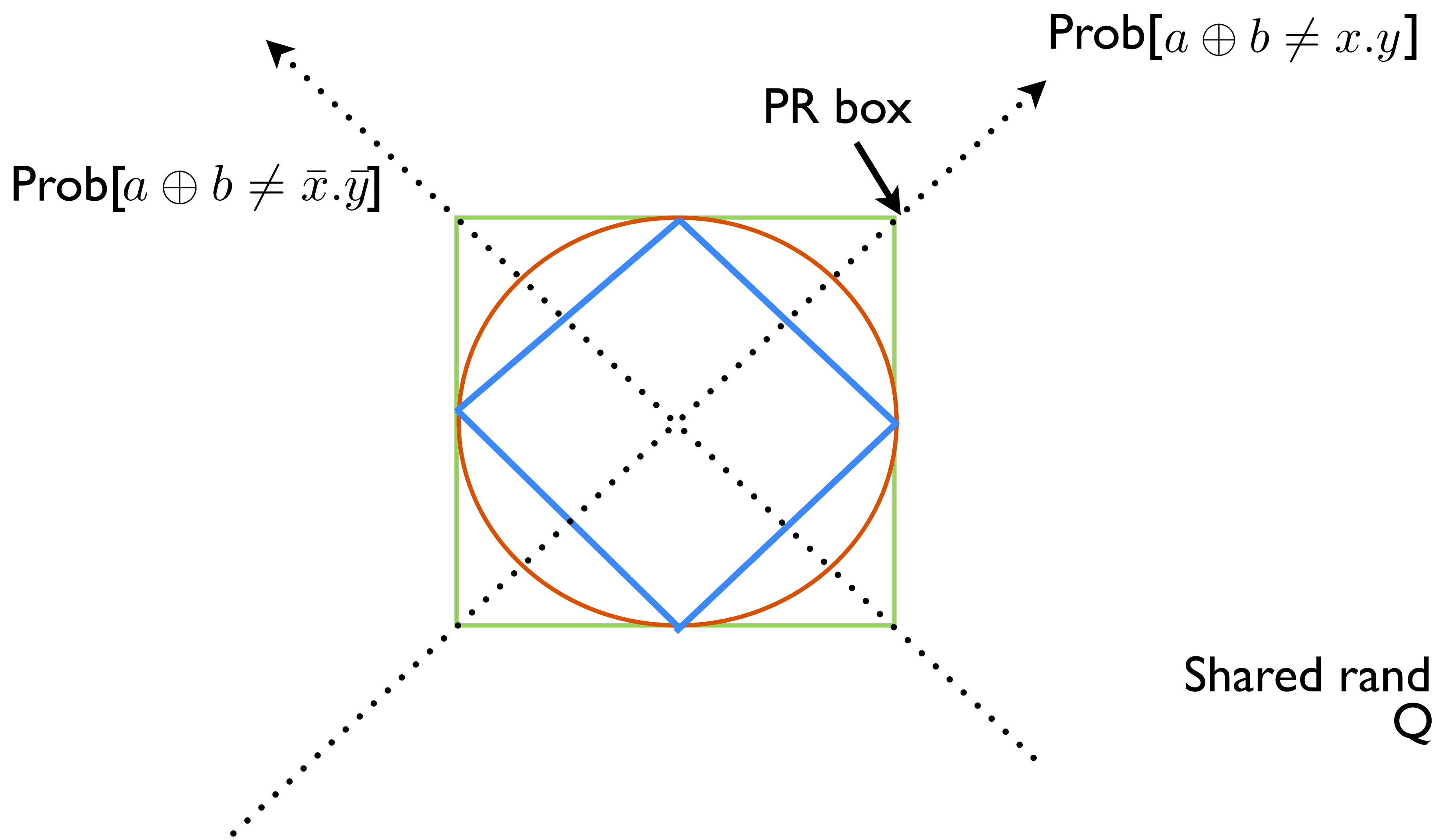
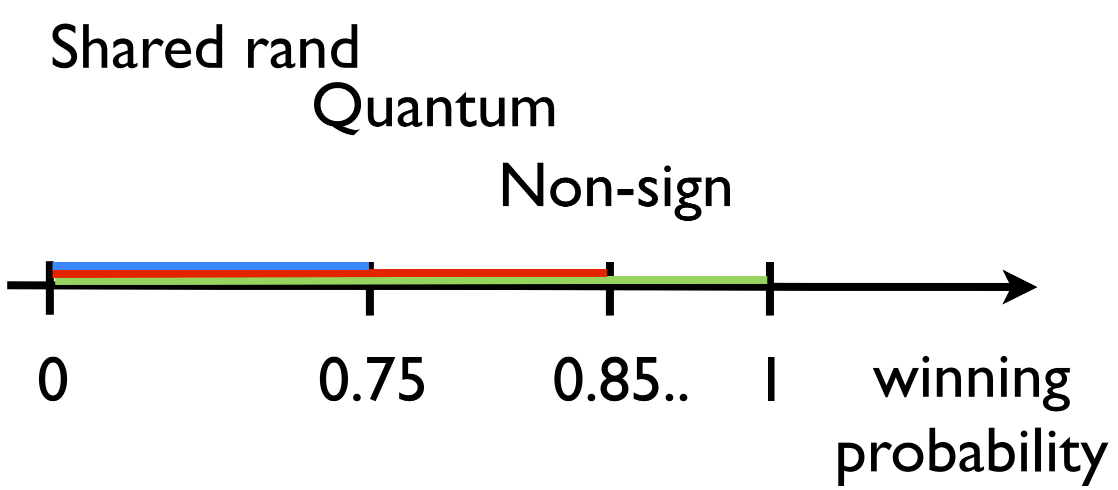
a=b for x=1, y=1

. They always win!

Comparison of correlations

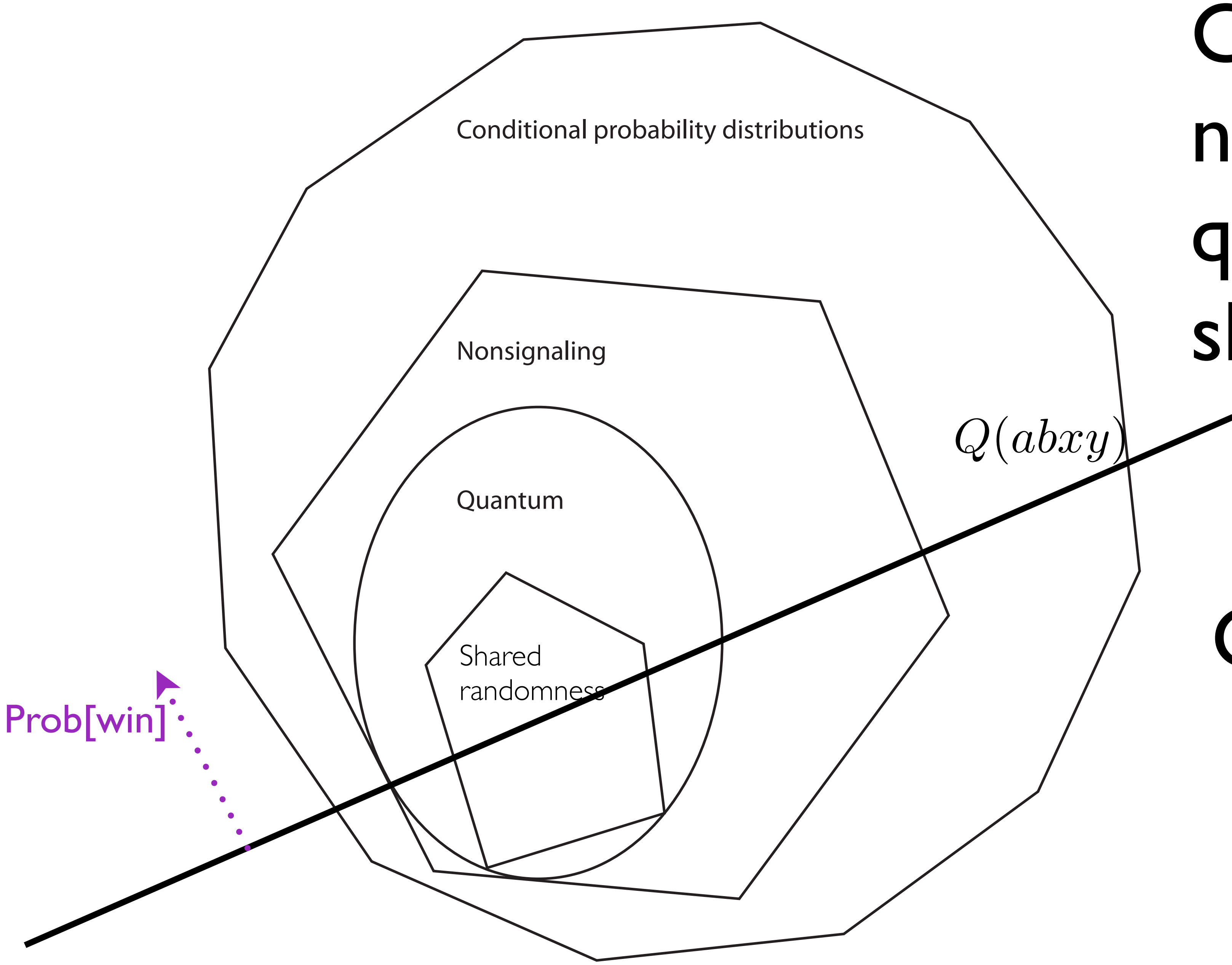




More parties? More questions? More answers?

active research field

Comparison of correlations



Convex sets:
non-signaling: polytope
quantum: semidefinite
shared rand: polytope

Game is hyperplane