Constructing correlations that do not admit of a noncontextual hidden variable model (Rob Spekkens)

Exercise 1. ***Contextuality: understanding through engineering***

This problem challenges you to devise correlations that cannot be explained by a noncontextual hidden variable model (throughout this problem, "noncontextual" refers to the *traditional* notion of a noncontextual model, i.e. one that is measurement-noncontextual and outcome-deterministic). Note that we are not going to worry about whether these correlations can be reproduced by quantum theory or not. Therefore, the solution to this problem makes no reference to the quantum formalism at all. The purpose is to try and clarify the content of the assumption of noncontextual hidden variable models by exploring fictitious worlds, i.e. worlds with experimental correlations that need not be described by a quantum or a classical theory.

Recall from the lecture the following simple example of such correlations, devised by Ernst Specker in 1960. Imagine a world wherein there is a system upon which one can implement three distinct measurements each of which has two outcomes. We will denote the measurements by M_1, M_2 and M_3 and take the outcome set for each to be $\{0,1\}$. Imagine that the following *pattern of joint measurability* applies to these measurements: any pair can be implemented jointly, but it is impossible to measure all three jointly. (Aside: this sort of pattern of joint measurability does not arise in quantum theory). The *correlations* are as follows: if any pair of distinct measurements are implemented jointly, then their outcomes are always anticorrelated – one is 0 and the other is 1. Call these the **Specker correlations.**

In a traditional noncontextual hidden variable model, the ontic state specifies an outcome for each measurement, that is, the ontic state prescribes a triple, X_1, X_2 and X_3, where X_i is the value that would be revealed by a measurement of M_i, regardless of which other measurement it is implemented jointly with. For instance, if the ontic state specifies that M_1 yields the value X_1 when it is measured jointly with M_2, then it specifies that M_1 must also yield the value X_1 when it is measured jointly with M_3.

There are many ways to see that there is no traditional noncontextual hidden variable model for the Specker correlations.

For instance, it suffices to note that there are eight possible value-assignments to (X_1, X_2, X_3), namely, $(0,0,0), (0,0,1), (0,1,0), \ldots (1,1,1)$. And for each of these, we can have anticorrelation for at most two of the three possible pairs of measurements that can be performed. For instance, $(0,1,0)$ yields anticorrelation for joint measurements of M_1 and M_2 and for joint measurements of M_2 and M_3, but *correlation* for joint measurements of M_1 and M_3.

Another way to see the impossibility of a traditional noncontextual model for these correlations is as follows. If $X_i^{(ij)}$ denotes an outcome of M_i when it is measured as part of the pair M_i and M_j, then the outcomes are assumed to satisfy the following constraints:

$$X_1^{(12)} = \left(X_2^{(12)} + 1\right) \mathrm{mod} 2, \tag{1}$$

$$X_1^{(13)} = \left(X_3^{(13)} + 1\right) \mathrm{mod} 2, \tag{2}$$

$$X_2^{(23)} = \left(X_3^{(23)} + 1\right) \mathrm{mod} 2. \tag{3}$$

A noncontextual hidden variable model specifies that

$$X_1^{(12)} = X_1^{(13)} \equiv X_1, \tag{4}$$
$$X_2^{(12)} = X_2^{(23)} \equiv X_2, \tag{5}$$
$$X_3^{(13)} = X_3^{(23)} \equiv X_3, \tag{6}$$

but given these equalities, Eqs. (1)-(3) cannot be satisfied.

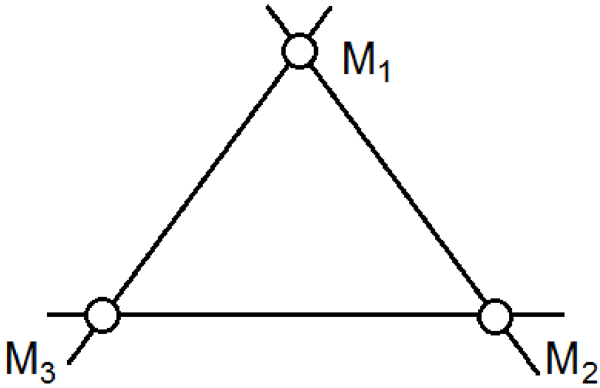

Figure 1: The pattern of joint measurablility for the Specker example

It is useful to represent such correlations by networks, where the different nodes represent the different measurements and the presence of a continuous line connecting a set of nodes represents the fact that the associated measurements can be implemented jointly. Fig. 1 illustrates the pattern of joint measurability for the Specker example. It is critical to note that the different continuous lines in the graphs represent different *counterfactual* possibilities for the joint measurement – in any given run of the experiment, only the set of measurements along **one** of the lines can be implemented. Different lines correspond to "complementary" possibilities. The correlations that hold among the outcomes of a set of measurements when these are implemented jointly are **not** illustrated in a figure of this sort, however one could achieve this by labeling the edges (as was done in class, where a dashed edge represented anticorrelation and a solid edge represented correlation).

At the end of this question, we provide many more examples of correlations that do not admit of a noncontextual hidden variable model, to give you an impression of their diversity. Go over the examples and make sure that you understand why these correlations cannot be explained with an outcome-deterministic measurement-noncontextual hidden variable model.

Your mission in this tutorial (should you choose to accept it) is to generate more examples of this phenomena. You may want to start with generalizations of the ones presented here, and think about families of examples that can be generated by such generalizations. Ultimately, however, the goal is for you to devise some examples that are **as different as possible** from the ones presented here (or seen in class). Be creative, talk to each other and share your ideas!

Just to be absolutely clear on what's being asked of you, here is what you must specify for each example you devise:

1) The nature of the measurements. How many measurements are there? For each measurement, how many outcomes does it have? (In all the examples we've provided, every

measurement has only two outcomes, but you shouldn't necessarily restrict yourself to such cases).

2) The pattern of joint measurability. Which subsets of measurements can be implemented jointly? (Note that in each of the examples presented in the gallery, the pattern of joint measurability is described by a diagram. These diagrams should be considered as a way to prime your ideas about possible such patterns, rather than as a constraint on the form in which you should present your examples. You might well find interesting examples that do not easily admit of a diagrammatic representation.)

3) The correlations. For each subset of measurements that can be implemented jointly, what is the probability distribution over the possible joint outcomes? (It may be sufficient to just specify certain features of this probability distribution – for instance, in the Specker correlations it was sufficient to say that the outcomes are anticorrelated without specifying the relative probability of $(0, 1)$ and $(1, 0)$.)

Finally, you must be able to show that there is no outcome-deterministic measurement-noncontextual hidden variable model for your example. Specifically, if there are N measurements, you must show that there is no joint value assignment to variables $(X_1, ..., X_N)$ that is consistent with the correlations you have posited.

Good luck!

For the extra-adventurous: If you are particularly pleased with one of your examples and you become curious about whether it might lead to a novel proof of the impossibility of a noncontextual hidden variable model **within quantum theory**, then to satisfy your curiosity, you may want to ask yourself the following questions: (1) Is the pattern of joint measurability posited in the example achievable in quantum theory? (For the Specker example above, it is not), (2) From your example, can you devise an inequality that must be respected by experimental statistics? For instance, you could ask: if one of the sets of measurements that can be implemented jointly is chosen uniformly at random from among all such sets, what is the maximum probability of generating the correct correlations within a noncontextual hidden variable model? (For the Specker parable, it is 2/3, for the CHSH game, it is 3/4, for Mermin's magic square, it is 5/6.) This upper bound on the maximum probability is an example of such an inequality. (3) Can you find a set of projection-valued measures representing your measurements and a quantum state such that the correlations these predict violate your inequality? If you get this far, and the example is new, the next step is to publish.

A gallery of examples of contextuality

Popescu-Rohrlich correlations

These correlations, although usually presented in proofs of nonlocality, are also an example of correlations that do not admit of a noncontextual hidden variable model. There are four measurements, which we denote M_1, M_2, M_3, M_4, each of which has two possible outcomes. The pattern of joint measurability is that we can measure any one of the following pairs: $\{M_1, M_3\}$, $\{M_2, M_3\}$, $\{M_1, M_4\}$, and $\{M_2, M_4\}$, but any other subset of two or more measurements cannot be measured jointly. The correlations are as follows: perfect correlation for $\{M_1, M_3\}$, $\{M_2, M_3\}$ and $\{M_1, M_4\}$, perfect anticorrelation for $\{M_2, M_4\}$. We see that there is no outcome-deterministic noncontextual hidden variable model because noncontextual values (drawn from

$\{0,1\}$) would need to satisfy $X_1 = X_3$, $X_2 = X_3$, $X_1 = X_4$ and $X_2 = (X_4 + 1) \bmod 2$, which is a contradiction. As a network, the correlations appear as follows:

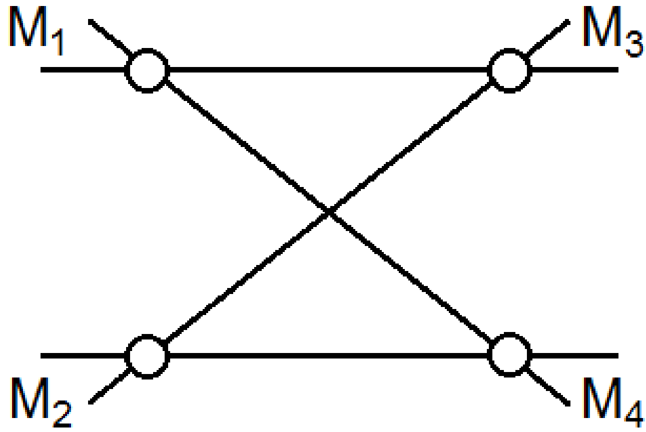

Figure 2: Joint measurability in the CHSH proof

Mermin's magic square

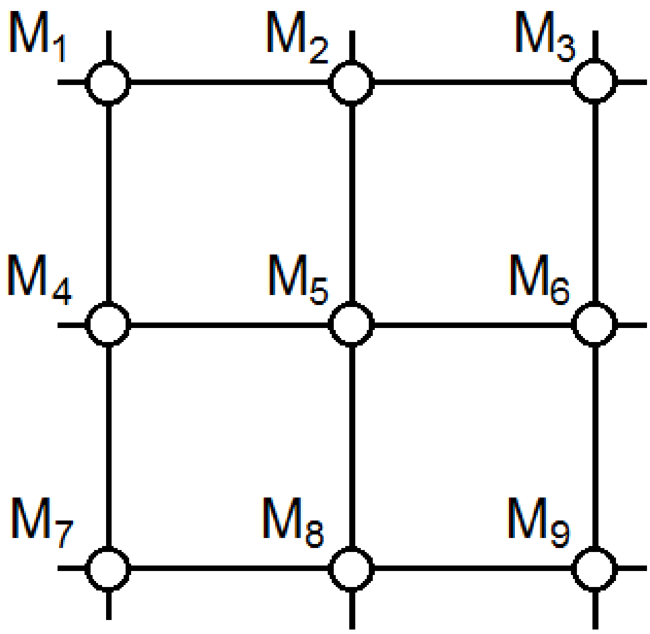

Figure 3: The joint-measureability of measurements in Mermin's magic square example

This example was also considered in lecture. There are nine two-outcome measurements, denoted $M_1, M_2, \ldots M_9$, for which only the triples corresponding to the rows and columns in fig. 3 can be measured jointly.

Adopt the convention wherein the outcomes of all the measurements are labeled by $\{+1, -1\}$. The correlations are as follows: for five out of the six triples of measurements, the product of the outcomes is 1, while for the last triple, the product is -1. (Equivalently, if we label the outcomes by $\{0, 1\}$, then five out of the six triples yield outcomes with total parity 0, while one yields outcomes with total parity 1.) It is not hard to see that no noncontextual values can reproduce these correlations.

Mermin's magic pentagram

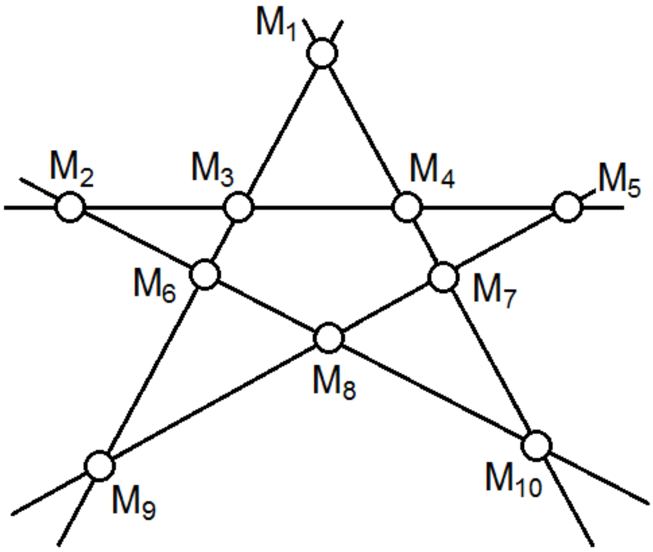

Figure 4: Mermin's magic pentagram example

In this example, also due to Mermin, there are ten two-outcome measurements, denoted M_1, M_2,... M_{10}, for which only the quadruples along one of the five lines shown in fig. 4 can be measured jointly.

As in the previous example, the outcomes of all the measurements are labeled by $\{+1, -1\}$. The correlations for this example are as follows: for four out of the five quadruples of measurements, the product of the outcomes is 1, while for the last quadruple, the product is -1. Again, it is not hard to see that no noncontextual values can reproduce these correlations.

GHZ correlations

In this example there are six possible two-outcome measurements, denoted $M_1, M_2, \ldots M_6$. The four lines shown in fig. 5 indicate which triples are jointly measurable. Sticking with our convention of labeling the outcomes of measurement by $\{+1, -1\}$, the correlations are as follows: for three of the four triples the product of outcomes is $+1$ while for the fourth it is -1. Again, it is straight-forward to verify that no noncontextual values can reproduce these correlations. This is the basis of the Greenberger, Horne and Zeilinger (GHZ) proof of nonlocality, although here we are presenting it as a proof of contextuality.

Hardy-type version of Specker's example

In this example, the pattern of joint measurability is the same as in the Specker example, depicted in fig. 1. The correlations, however, are different. In a joint measurement of M_1 and M_2, it is assumed that the probability distribution over the four possible outcomes (denoted (X_1,X_2)) assigns non-zero probability to $(0,0), (0,1)$ and $(1,0)$, and zero probability to $(1,1)$. In this case, we can be sure that if, in a joint measurement of M_1 and M_2, M_1 yields outcome $X_1 = 1$, then M_2 must yield outcome $X_2 = 0$. The same probability distribution over outcomes is assumed to arise for a joint measurement of M_3 and M_1, while for a joint measurement of M_2 and M_3, it is the $(0,0)$ outcome that is assigned zero probability, while the others are assigned

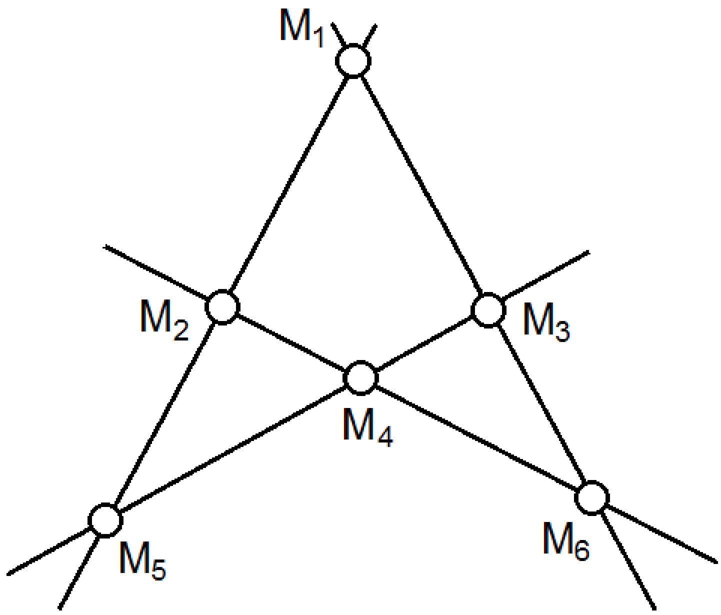

Figure 5: Joint measurability in the GHZ proof

nonzero probability. Finally, it is assumed that when M_1 is measured with M_2, the outcome $X_1 = 1$ occurs with nonzero probability. It is not too difficult to see that these correlations cannot arise in a noncontextual hidden variable model. In order to explain the fact that $X_1 = 1$ occurs with nonzero probability in a measurement of M_1, such a model must posit at least one ontic state which fixes the outcome of M_1 to be $X_1 = 1$. However, for this ontic state, the outcome of M_2 would have to be $X_2 = 0$ because otherwise a joint measurement of M_1 and M_2 would sometimes yield the $(1, 1)$ outcome. Furthermore, for this ontic state, the outcome of M_3 would have to be $X_3 = 1$ because otherwise a joint measurement of M_2 and M_3 would sometimes yield the $(0, 0)$ outcome. Finally, for this ontic state, the outcome of M_1 would have to be $X_1 = 0$ because otherwise a joint measurement of M_3 and M_1 would sometimes yield the $(1, 1)$ outcome. But by assumption $X_1 = 1$ for this ontic state, so we have arrived at a contradiction. We can summarize the example as follows. The following implications hold: $X_1 = 1 \implies X_2 = 0$, $X_2 = 0 \implies X_3 = 1$, $X_3 = 1 \implies X_1 = 0$ (see fig. 6 for an illustration of these implications). Furthermore, it is sometimes the case that $X_1 = 1$, which by the chain of implications yields $X_1 = 0$, a contradiction.

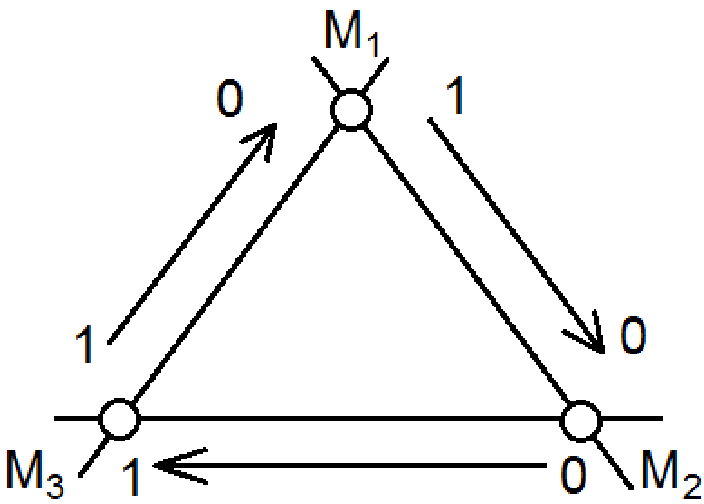

Figure 6: Hardy-type version of Specker's example