Exercise 1. ***Entropic relations***

(a) Prove

$$H(Z^A|B)_\rho + H(X^A|B)_\rho \geq \log_2 d + H(A|B)_\rho \tag{1}$$

where ρ is over systems AB and d is the dimension of system A.

Hint. First show that for any system over $A'B'AB$ we have

$$H(A'|AB) + H(B'|AB) \geq H(A'B'|AB) \;. \tag{2}$$

Then define

$$\Omega^{A'B'AB} = \frac{1}{d^2}\sum_{jk} P_j^{A'} \otimes P_k^{B'} \otimes \rho_{jk}^{AB}$$

where $\rho_{jk}^{AB} = X_j^A Z_k^A \rho Z_k^{\dagger A} X_j^{\dagger A}$, $P_i = |i\rangle\langle i|$ and $dim(A') = dim(B') = d$ and show that

$$H(A'|AB)_\Omega + H(B'|AB)_\Omega \geq H(A'B'|AB)_\Omega$$

implies

$$H(Z^A|B)_\rho + H(X^A|B)_\rho \geq log_2 d + H(A|B)_\rho \;.$$

(b) Prove that if $H(X^A|B)_\rho = 0$ or $H(Z^A|E)_\rho = 0$ then

$$H(Z^A|B)_\rho + H(X^A|E)_\rho = \log_2 d \;.$$

As in the lecture, consider the following states:

$$|\psi\rangle^{ABR} = \sum_z \sqrt{p_z}|z\rangle^A \otimes |\varphi_z\rangle^{BR}$$
$$|\psi'\rangle^{ABCR} = \sum_z \sqrt{p_z}|z\rangle^A|z\rangle^C \otimes |\varphi_z\rangle^{BR}$$

(c) Derive the conditions

$$H(Z^A|B)_\psi \leq \varepsilon_1^2$$
$$H(Z^A|R)_\psi \geq \log_2 d - \varepsilon_2^2$$

from

$$H(X^A|RC)_{\psi'} \geq \log_2 d - \varepsilon_1^2$$
$$H(Z^A|R)_\psi \geq \log_2 d - \varepsilon_2^2$$

using the previous item.