

Exercise 1. Pusey–Barrett–Rudolph argument for non-orthogonal states

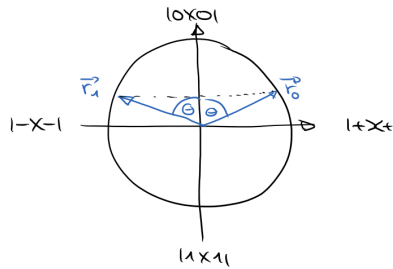
In this exercise you will generalize the argument given in the lecture to arbitrary non-orthogonal states by following the appendix of the paper by Pusey, Barrett and Rudolph (arXiv:1111.3328). For this, suppose that you are given n copies of a device that prepares a quantum system in either the state $|\psi_0\rangle$ or the state $|\psi_1\rangle$. Thus there are 2^n possible preparations, $|\psi_{\vec{x}}\rangle = |\psi_{x_1}\rangle \otimes \dots \otimes |\psi_{x_n}\rangle$, where \vec{x} is a binary string with $x_i = 0/1$ if the i -th system is prepared in state $|\psi_{0/1}\rangle$. The challenge is to find a joint measurement of the n systems such that each measurement outcome has probability zero for (at least) one of the preparations.

(a) Show that (up to global phases) there always exist orthonormal vectors $|0\rangle, |1\rangle$ such that

$$|\psi_0\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} |1\rangle, \quad |\psi_1\rangle = \cos \frac{\theta}{2} |0\rangle - \sin \frac{\theta}{2} |1\rangle,$$

with $0 < \theta \leq \frac{\pi}{2}$.

Solution. We can always rotate the Bloch sphere (i.e., the basis vectors $|0\rangle$ and $|1\rangle$) such that



Then it is clear that the states $|\psi_i\rangle\langle\psi_i|$ (whose Bloch vectors we have denoted by r_i) have the desired form (with θ as in the figure).

We shall think of $|0\rangle$ and $|1\rangle$ defining the computational basis of a qubit, and write $|\vec{x}\rangle = |x_1\rangle \otimes \dots \otimes |x_n\rangle$ for the corresponding product basis.

(b) Consider the following measurement procedure: First, apply the unitary $Z_\beta = \begin{pmatrix} 1 & \\ & e^{i\beta} \end{pmatrix}$ to each qubit. Second, apply the unitary R_α which maps $|0\dots 0\rangle \mapsto e^{i\alpha}|0\dots 0\rangle$ and acts as the identity on the orthogonal complement. Third, apply a Hadamard gate $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to each qubit. Finally, measure each qubit in the computational basis. Show that the probability of outcome $|\vec{x}\rangle$ given the initial state $|\psi_{\vec{x}}\rangle$ as predicted by quantum mechanics is equal to

$$\frac{1}{2^n} \left(\cos \frac{\theta}{2} \right)^{2n} \left| e^{i\alpha} - 1 + \left(1 + e^{i\beta} \tan \frac{\theta}{2} \right)^n \right|^2.$$

Solution. The desired probability is the *modulus squared* of

$$\begin{aligned} & \langle \vec{x} | H^{\otimes n} R_\alpha Z_\beta^{\otimes n} | \psi_{\vec{x}} \rangle \\ &= \frac{1}{\sqrt{2^n}} \left(\sum_{\vec{y}} (-1)^{\vec{x} \cdot \vec{y}} \langle \vec{y} | \right) R_\alpha \bigotimes_i \left(\cos \frac{\theta}{2} |0\rangle + (-1)^{x_i} e^{i\beta} \sin \frac{\theta}{2} |1\rangle \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2^n}} \left(e^{i\alpha} \langle 0 \dots 0 | + \sum_{\vec{y} \neq 0} (-1)^{\vec{x} \cdot \vec{y}} \langle \vec{y} | \right) \otimes_i \left(\cos \frac{\theta}{2} |0\rangle + (-1)^{x_i} e^{i\beta} \sin \frac{\theta}{2} |1\rangle \right) \\
&= \frac{1}{\sqrt{2^n}} \left(e^{i\alpha} \left(\cos \frac{\theta}{2} \right)^n + \sum_{\vec{y} \neq 0} (-1)^{\vec{x} \cdot \vec{y}} (-1)^{\vec{x} \cdot \vec{y}} \left(\cos \frac{\theta}{2} \right)^{n - \|\vec{y}\|_1} \left(e^{i\beta} \sin \frac{\theta}{2} \right)^{\|\vec{y}\|_1} \right) \\
&= \frac{1}{\sqrt{2^n}} \left(e^{i\alpha} \left(\cos \frac{\theta}{2} \right)^n + \sum_{y=1, \dots, n} \binom{n}{y} \left(\cos \frac{\theta}{2} \right)^{n-y} \left(e^{i\beta} \sin \frac{\theta}{2} \right)^y \right) \\
&= \frac{1}{\sqrt{2^n}} \left((e^{i\alpha} - 1) \left(\cos \frac{\theta}{2} \right)^n + \sum_{y=0, \dots, n} \binom{n}{y} \left(\cos \frac{\theta}{2} \right)^{n-y} \left(e^{i\beta} \sin \frac{\theta}{2} \right)^y \right) \\
&= \frac{1}{\sqrt{2^n}} \left((e^{i\alpha} - 1) \left(\cos \frac{\theta}{2} \right)^n + \left(\cos \frac{\theta}{2} + e^{i\beta} \sin \frac{\theta}{2} \right)^n \right) \\
&= \frac{1}{\sqrt{2^n}} \left(\cos \frac{\theta}{2} \right)^n \left(e^{i\alpha} - 1 + \left(1 + e^{i\beta} \tan \frac{\theta}{2} \right)^n \right)
\end{aligned}$$

(hence the overall phases which we had ignored in (a) indeed do not matter).

- (c) Show that for large enough n , the phases α, β can be chosen such that these probabilities are all zero.

Solution. We would like to find n, α and β such that

$$e^{i\alpha} - 1 + \left(1 + e^{i\beta} \tan \frac{\theta}{2} \right)^n = 0.$$

Of course, it suffices to find n and β such that

$$|1 - \left(1 + e^{i\beta} \tan \frac{\theta}{2} \right)^n| = 1. \quad (\text{S.1})$$

Geometrically, $f_n(\beta) = 1 - \left(1 + e^{i\beta} \tan \frac{\theta}{2} \right)^n$ defines a curve in the complex plane for any fixed n , and the condition (S.1) amounts to showing that this curve intersects the unit circle. Note that $\tan \frac{\theta}{2} \in (0, 1]$ since $\theta \in (0, \frac{\pi}{2}]$ (see picture), and therefore

$$f_n(\pi) = 1 - \left(1 - \tan \frac{\theta}{2} \right)^n \in (0, 1]$$

is always contained in the unit disk. On the other hand, for large enough $n \gg 1$

$$f_n(0) = 1 - \left(1 + \tan \frac{\theta}{2} \right)^n < -1.$$

For such n , continuity of $f_n(\beta)$ ensures that there exists β with $|f_n(\beta)| = 1$.

- (d) Conclude that, under the assumptions 1–3 stated in the lecture, there does not exist a physical state z compatible with both $|\psi_0\rangle$ and $|\psi_1\rangle$.

Solution. This works precisely as you saw in the lecture.

Exercise 2. *Reality of the wave function from different assumptions*

In this exercise, your task is to understand an alternative argument due to Colbeck and Renner (arXiv:1111.6597), which derives the reality of the wave function from a different set of assumptions. As in the lecture, we consider a system prepared in a state described by a wave function Ψ ; the experimenter then chooses a measurement setting A and records the measurement outcome X . Mathematically, Ψ, A , and X are modelled by random variables on some underlying

probability space. Let $\Gamma \ni \Psi$ be a collection of random variables on the same probability space, modeling all information that is in principle available before the measurement setting is chosen. Technically, we shall assume that *measurement settings can be chosen freely*; in particular, this implies that $\mathbb{P}_{A|\Gamma} = \mathbb{P}_A$. We shall also assume that *quantum theory is correct*, so that e.g. $\mathbb{P}_{X|A\Psi}$ is given by the predictions of quantum mechanics.

Under these assumptions, it has been shown that the wave function Ψ is complete for the description of this system (arXiv:1005.5173). Here and in the following, a subset of random variables $\Gamma_0 \subseteq \Gamma$ is said to be *complete* for the description of the system if $\mathbb{P}_{X|\Gamma A} = \mathbb{P}_{X|\Gamma_0 A}$, i.e. if

$$\Gamma \rightarrow (\Gamma_0, A) \rightarrow X$$

is a Markov chain.

- (a) Compare this notion of completeness with the one discussed in last semester's quantum information theory lecture. (*This part of the exercise is optional.*)

Let us now consider another subset of random variables $Z \subseteq \Gamma$ (a "list of elements of reality") that is also complete for the description of the system.

- (b) Show that

$$\mathbb{P}_{X|Z=z, A=a} = \mathbb{P}_{X|\Psi=\psi, A=a}$$

whenever $\mathbb{P}(Z = z, \Psi = \psi) > 0$ and $\mathbb{P}(A = a) > 0$.

Solution. It follows from the completeness of Z and Ψ that we have two Markov chains

$$\begin{aligned} \Gamma \ni \Psi &\rightarrow (Z, A) \rightarrow X, \\ \Gamma \supseteq Z &\rightarrow (\Psi, A) \rightarrow X. \end{aligned}$$

Therefore,

$$\mathbb{P}_{X|ZA} = \mathbb{P}_{X|\Psi ZA} = \mathbb{P}_{X|\Psi A},$$

or, equivalently,

$$\mathbb{P}_{X|Z=z, A=a} = \mathbb{P}_{X|\Psi=\psi, A=a}$$

for all z, ψ and a for which

$$\mathbb{P}(Z = z, \Psi = \psi, A = a) = \mathbb{P}(Z = z, \Psi = \psi) \mathbb{P}(A = a) > 0.$$

(the last equality follows from the assumption of free measurement settings).

- (c) Conclude that the wave function Ψ is determined uniquely by Z . In this sense, the system's wave function is in one-to-one correspondence with its elements of reality.

Solution. Suppose ψ and ψ' are both compatible with z , i.e.

$$\mathbb{P}(Z = z, \Psi = \psi) > 0, \quad \mathbb{P}(Z = z, \Psi = \psi') > 0.$$

Then (b) implies that

$$\mathbb{P}_{X|\Psi=\psi, A=a} = \mathbb{P}_{X|\Psi=\psi', A=a}$$

whenever $\mathbb{P}(A = a) > 0$. Clearly, there exists a measurement setup for which this implies that $\psi = \psi'$ (e.g., the projective measurement $\{|\psi\rangle\langle\psi|, \mathbf{1} - |\psi\rangle\langle\psi|\}$). Thus there exists at most a single wave function $\Psi = \psi$ compatible with each $Z = z$, i.e. Ψ is uniquely determined by Z .