Basics of Lie theory

Classification of Lie Algebras

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The Matrix group SO(3)

Consider the Matrix group

$$\mathrm{SO}(3) = \{A \in \mathrm{Mat}(3, \mathbb{R}) \mid A^T A = \mathbb{1}, \det(A) = 1\}$$

Define the **Lie algebra of SO(3)** as

$$\mathfrak{so}(3) = \{\dot{\gamma}(0) \mid \gamma : (-\varepsilon, \varepsilon) \to \mathrm{SO}(3), \gamma(0) = \mathbb{1}\}$$

Claim

$$\mathfrak{so}(3) = \{A \in \mathrm{Mat}(3, \mathbb{R}) \mid A^T + A = 0\}$$

Proof of the Claim:

"$\subset$" Consider γ as in the definition of the Lie algebra. Then

$$\gamma(t)^T\gamma(t) = \mathbb{1} \quad \forall t \in [0, \varepsilon)$$

By differentiation

$$\dot{\gamma}(t)^T\gamma(t) + \gamma(t)^T\dot{\gamma}(t) = 0$$
$$\stackrel{t=0}{\Rightarrow} \dot{\gamma}(0)^T + \dot{\gamma}(0) = 0$$

"$\supset$" Let $A \in \mathrm{Mat}(3,\mathbb{R})$ st. $A^T + A = 0$. In particular $\mathrm{Tr}(A) = 0$. Define

$$\begin{aligned} \gamma : \mathbb{R} &\to \mathrm{Mat}(3,\mathbb{R}) \\ t &\mapsto \exp(At) \end{aligned}$$

Note that

1. $\gamma(0) = \mathbb{1}$
2. $\det(\gamma(t)) = \exp(t\,\mathrm{Tr}(A)) = 1$
3. $\gamma(t)^T\gamma(t) = \exp(-At)\exp(At) = \mathbb{1}$
4. $\dot{\gamma}(0) = A$ $\square$

Lie groups

Definition

A Lie group G is a set that has compatible structures of a smooth manifold and of a group. Compatible means that group multiplication and inversion are smooth maps i.e. the maps $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth

A **Matrix Lie group** is a Lie group that is contained in $\mathrm{GL}(n, \mathbb{K})$ for some n and field K. Let $n \in \mathbb{N}$. Then the following groups are Lie groups

- $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$
- $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{C})$
- $\mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), SU(n)$
- The symplectic groups $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{Sp}(2n, \mathbb{C})$
- The group B_n of upper-triangular matrices

Construction of the Lie algebra

Consider the action of the Lie group G on itself by conjugation

$$\Psi : G \rightarrow Aut(G)$$
$$g \mapsto \psi_g$$

where

$$\psi_g(h) = ghg^{-1} \quad \forall h \in G$$

Note that the neutral element e gets mapped to itself. Consider now for $g \in G$ the map

$$Ad(g) = (d\psi_g)_e : T_eG \rightarrow T_eG$$

Thus

$$Ad : G \longrightarrow Aut(T_e G)$$

Taking the differential map of Ad at the unity we get a map in the tangent spaces

$$ad : T_e G \longrightarrow End(T_e G)$$

This implies a bilinear map $T_e G \times T_e G \longrightarrow T_e G$ called the **Lie bracket** by

$$[X, Y] := ad(X)(Y)$$

Theorem

The Lie bracket fulfills

- $[X, Y] = -[Y, X]$ for all $X, Y \in T_e G$
- the Jacobi identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

for all $X, Y, Z \in T_e G$

The **Lie algebra associated to the Lie group** G is $T_e G$ together with the Lie bracket on $T_e G$, we write $\mathfrak{g}$. A vectorspace $\mathfrak{g}$ together with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the conditions in the theorem above is called a **Lie algebra**.

Homomorphisms of Lie groups and Lie algebras

Definition

Let G,H be Lie groups and $\mathfrak{g}, \mathfrak{h}$ a Lie algebras

- A Lie group homomorphism $\rho : G \longrightarrow H$ is a smooth map such that $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$.
- A Lie algebra homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a linear map, such that $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$ for all $X, Y \in \mathfrak{g}$.

A **representation of a Lie group** G is a Lie group homomorphism mapping to GL(V), where V is some vector space.
A **representation of a Lie algebra** $\mathfrak{g}$ is a Lie algebra homomorphism mapping to $\mathfrak{gl}(V) = End(V)$.

Fact

- Let G a Lie group and $\mathfrak{g}$ its Lie algebra. If G is connected, it is possible to generate the whole Lie group using $\mathfrak{g}$ only.
- Let G,H Lie groups and $\mathfrak{g}, \mathfrak{h}$ its Lie algebras. If G is simply connected, the Lie group homomorphisms from G to H are in one-to-one correspondence to the Lie algebra homomorphisms from $\mathfrak{g}$ to $\mathfrak{h}$.

Examples of Lie algebras

Matrix Lie groups $\longrightarrow$ Matrix Lie algebras.
Some complex Matrix Lie algebras:

- $\mathfrak{gl}_n\mathbb{C} = \mathrm{End}(\mathbb{C}^n)$ (or more generally $\mathfrak{gl}(V)$ for V vector space)
- $\mathfrak{sl}_n\mathbb{C} = \{A \in \mathrm{Mat}(n, \mathbb{C}) \mid \mathrm{Tr}(A) = 0\}$
- $\mathfrak{sp}_{2n}\mathbb{C} = \{A \in \mathrm{Mat}(2n, \mathbb{C}) \mid MA + A^T M = 0\}$ where

$$M = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$$

- $\mathfrak{so}_{2n}\mathbb{C}$. As above, but with $M = \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}$
- $\mathfrak{so}_{2n+1}\mathbb{C}$. With $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathbb{1}_n \\ 0 & \mathbb{1}_n & 0 \end{pmatrix}$

Lie algebras - basic notions

A subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$, that is closed under the Lie bracket (i.e. $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$) is called a **Lie subalgebra**.

Definition

1. A Lie subalgebra $\mathfrak{h}$ is an **ideal** if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.
2. A Lie algebra $\mathfrak{g}$ is **abelian** if $[\mathfrak{g}, \mathfrak{g}] = 0$.
3. A non-abelian Lie algebra $\mathfrak{g}$ that does not contain any non-trivial ideal, is called **simple**.
4. A Lie algebra $\mathfrak{g}$ that does not contain any abelian ideal is called **semisimple**.

Example 1: The center $Z(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \ \forall Y \in \mathfrak{g}\}$ is an ideal. The center of a semisimple Lie algebra contains only 0.
Example 2: $\mathfrak{sl}_n\mathbb{C} \subset \mathfrak{gl}_n\mathbb{C}$ is a non-abelian ideal.

The adjoint map

Let $\mathfrak{g}$ be a complex Lie algebra in what follows. The **adjoint map** at $X \in \mathfrak{g}$ is

$$\begin{aligned} \mathrm{ad}_X : \mathfrak{g} &\to \mathfrak{g} \\ Y &\mapsto [X, Y] \end{aligned}$$

One can show that

$$\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$$

Thus ad is a representation of $\mathfrak{g}$ on itself $\to$ **adjoint representation**.

Example: a basis for $\mathfrak{sl}_2(\mathbb{C})$

We consider the following basis of $\mathfrak{sl}_2(\mathbb{C})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then

$$[H, X] = 2X, \; [H, Y] = -2Y, \; [X, Y] = H$$

It can easily be shown that $\mathfrak{sl}_2(\mathbb{C})$ is simple using the relations above.

Cartan subalgebra

Let $\mathfrak{g}$ a semisimple (finite) Lie algebra. Consider a maximal subset of $\mathfrak{g}$ consisting of linearly independent, commuting elements, st. for each element H ad_H is diagonalizable (i.e. H is **ad-diagonalizable**). The subalgebra spanned by these elements is called a **Cartan subalgebra**, denoted by $\mathfrak{h}$. Note that

- The Cartan subalgebra is unique up to automorphisms of $\mathfrak{g}$.
- The Cartan subalgebra is a maximal abelian subalgebra consisting of simultaneously ad-diagonalizable elements b.c.

$$[\mathrm{ad}_{H_1}, \mathrm{ad}_{H_2}] = \mathrm{ad}_{[H_1, H_2]} = 0 \quad \forall H_1, H_2 \in \mathfrak{h}$$

- $\mathfrak{h}$ is non trivial.

Cartan decomposition

$\rightarrow$ action of $\mathfrak{h}$ on $\mathfrak{g}$ by adjoint representation (diagonalizable!). This yields the **Cartan decomposition**

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha$$

where g_α are eigenspaces of the action of $\mathfrak{h}$. For $H \in \mathfrak{h},\ X \in \mathfrak{g}_\alpha$ we have

$$\mathrm{ad}_H(X) = [H, X] = \alpha(H)X$$

$\rightarrow \alpha \in \mathfrak{h}^*$, called **roots**. $\mathfrak{g}_\alpha$ are the **root spaces**

Action of $\mathfrak{g}_\alpha$ on $\mathfrak{g}$

Claim

In the adjoint representation $\mathfrak{g}_\alpha : \mathfrak{g}_\beta \longrightarrow \mathfrak{g}_{\alpha+\beta}$

Proof: Let $X_\alpha \in \mathfrak{g}_\alpha, X_\beta \in \mathfrak{g}_\beta$ and $H \in \mathfrak{h}$. Then

$$\begin{aligned}[H,[X_\alpha,X_\beta]] &= -[X_\beta,[H,X_\alpha]] - [X_\alpha,[X_\beta,H]] \\ &= -\alpha(H)[X_\beta,X_\alpha] + \beta(H)[X_\alpha,X_\beta] \\ &= (\alpha+\beta)(H)[X_\alpha,X_\beta] \quad \square\end{aligned}$$

We will denote the set of roots by R.

On roots and root spaces

Proposition

Let $\mathfrak{g}$ a semisimple, complex, finite-dim. Lie algebra. Let $\mathfrak{h}$ a Cartan subalgebra. Consider the Cartan-decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha$$

Then

- The roots span the dual space $\mathfrak{h}^*$.
- Every root space is one dimensional.
- The only multiples of a root α, which are roots are $\pm\alpha$.

A basis of $\mathfrak{g}$ consisting of a basis of $\mathfrak{h}$ and of elements spanning $\mathfrak{g}_\alpha$ is called a **Cartan-Weyl basis**.

Remark

We can show that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$, $[[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \mathfrak{g}_\alpha] \neq 0$. Thus

$$s_\alpha := \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \simeq \mathfrak{sl}_2\mathbb{C}$$

We can thus choose $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$ and set $H_\alpha = [X_\alpha, Y_\alpha] \in \mathfrak{h}$, such that the usual commutation relations of $\mathfrak{sl}_2\mathbb{C}$ hold i.e.

$$[H_\alpha, X_\alpha] = 2X_\alpha, [H_\alpha, Y_\alpha] = -2Y_\alpha, H_\alpha = [X_\alpha, Y_\alpha]$$

In particular $\alpha(H_\alpha) = 2$.

It is possible to "build up" the Cartan subalgebra with elements $\{H_\alpha\}_{\alpha\in R}$. In fact we can choose a subset of R st. the above elements form a basis.

Proposition

There are elements $\{H_\alpha\}_{\alpha\in R}$ spanning $\mathfrak{h}$ such that $\beta(H_\alpha)$ is an integer for every $\alpha, \beta \in R$ and $\alpha(H_\alpha) = 2$.

The Killing form

For $X, Y \in \mathfrak{g}$ we define the Killing form as

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$$

Note that B is a linear map

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

It also clear, by definition of B, that B is symmetric.

Nondegeneracy of the Killing form

Proposition

The Killing form is positive definite on the real subspace of $\mathfrak{h}$ spanned by $\{H_\alpha\}_\alpha$.

Proposition

$\mathfrak{g}$ is semisimple iff its Killing form is nondegenerate.

Idea of the Proof: "$\Rightarrow$" Show that the kernel of B is an ideal. "$\Leftarrow$" Show that if I is an ideal, then $I^\perp$ is also an ideal.

Killing form on $\mathfrak{h}^*$

Remark

The nondegeneracy of the bilinear form (on the real subspace spanned by $\{H_\alpha\}_\alpha$) supplies an isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ under which

$$T_\alpha := 2H_\alpha / B(H_\alpha, H_\alpha) \mapsto \alpha$$

The Killing form on $\mathfrak{h}^*$ is defined by

$$B(\alpha, \beta) = B(T_\alpha, T_\beta)$$

for two roots $\alpha, \beta \in R$ (pos.def. on the subspace spanned by R). By definition

$$\beta(H_\alpha) = \frac{2B(\beta, \alpha)}{B(\alpha, \alpha)}$$

The Weyl group

Proposition

For any $\alpha \in R$ the map (an involution)

$$\begin{aligned} W_\alpha : \mathfrak{h}^* &\longrightarrow \mathfrak{h}^* \\ \beta &\mapsto \beta - \beta(H_\alpha)\alpha \end{aligned}$$

leaves R invariant.

The **Weyl group** is the group generated by the set of automorphisms $\{W_\alpha\}_{\alpha\in R}$. By the above the set of roots R is invariant under the Weyl group.

Since

$$W_\alpha(\beta) = \beta - \frac{2B(\beta,\alpha)}{B(\alpha,\alpha)}\alpha$$

W_α corresponds to a reflection in the hyperplane

$$\Omega_\alpha = \{\beta \in \mathfrak{h}^* : B(\beta,\alpha) = 0\}$$

Ordering of the roots

Pick a hyperplane in $\mathfrak{h}^*$ such that no point of the lattice spanned by R is contained and call by convention the points on one side the plane **positive** and on the other negative. A positive root is called **simple** if it cannot be written as a sum of two positive roots. E.g.

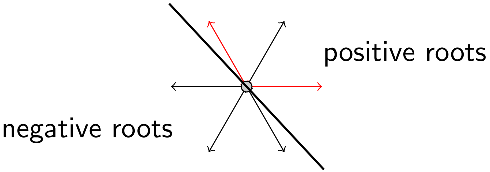


Figure: Root system of $\mathfrak{sl}_3\mathbb{C}$, splitting of the space by the thick line, simple roots in red.

Angles between roots

Denote by $\mathbb{E}$ the real subspace of $\mathfrak{h}^*$ spanned by the roots together with the scalar product given by the Killing form (denoted simply by $(\cdot,\cdot)$. Recall: $\forall \alpha, \beta \in R$

$$n_{\beta\alpha} := \frac{2B(\beta,\alpha)}{B(\alpha,\alpha)} = \beta(H_\alpha) \in \mathbb{Z}$$

If θ is the angle between α and β, then

$$n_{\beta\alpha} = 2\cos(\theta)\frac{||\beta||}{||\alpha||}$$

Thus

$$n_{\beta\alpha} n_{\alpha\beta} = 4\cos^2(\theta) \leq 4$$

Angles between roots

Hence $4\cos^2(\theta)$ is an integer. The allowed angles in $[0, \pi)$ are $\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$.

Example: Assume $|n_{\beta\alpha}| \geq |n_{\alpha\beta}|$ and $\theta = \frac{\pi}{6}$ for instance. Then $\cos(\theta) = \frac{\sqrt{3}}{2}$ and $n_{\beta\alpha} n_{\alpha\beta} = 3$. Hence $n_{\beta\alpha} = 3$ and $n_{\alpha\beta} = 1$ $\Rightarrow \frac{||\beta||}{||\alpha||} = \sqrt{3}$.

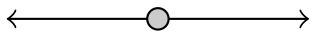
Examples of root systems

We call

$$r := \dim_{\mathbb{R}} \mathbb{E} = \dim_{\mathbb{C}} \mathfrak{h}$$

the **rank** of the Lie algebra.

rank 1 There is exactly one possible root system that can be drawn

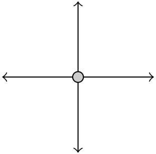
(A_1) 

This is precisely the root system of $\mathfrak{sl}_2\mathbb{C}$.

Examples of root systems

<u>**rank 2**</u> There are 4 different root system in 2 dimensions.

(A_1)x(A_1)



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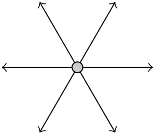
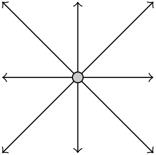
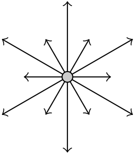


(B_2)



(G_2)



Further symmetries of the root system

Recall: a Lie algebra is *simple* if it is non-abelian and contains no non-trivial ideals.

Lemma

A semisimple Lie algebra is simple iff its root system is irreducible i.e. cannot be written as a direct sum of two root systems.

Also recall that a simple root is a root that cannot be written as a sum of two positive roots. One can show that:

- If α, β simple, then neither $\alpha - \beta$ nor $\beta - \alpha$ are roots.
- The angle between two simple roots cannot be acute.
- The simple roots are linearly independent and span $\mathbb{E}$. Every positive root can be uniquely written as a non-negative integral linear combination of simple roots.

Dynkin diagrams

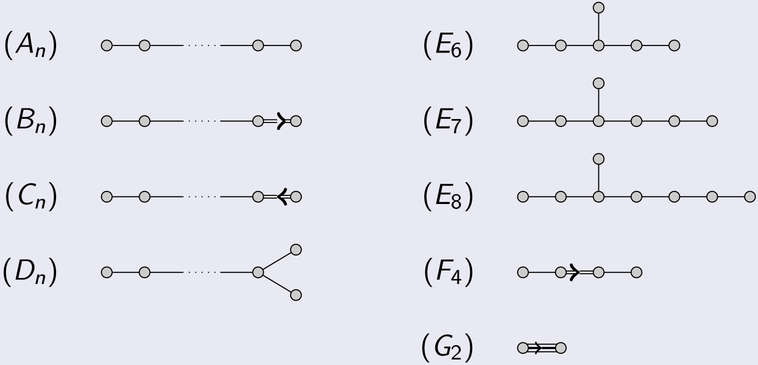
The **Dynkin diagram** of a root system is drawn as follows.

- Every simple root is represented by a node $\circ$.
- Two simple roots are connected in the following way
  - not connected, if $\theta = \frac{\pi}{2}$
  - one line, $\theta = \frac{2\pi}{3}$
  - two lines and an arrow pointing from the longer to the shorter root, if $\theta = \frac{3\pi}{4}$.
  - three lines and an arrow pointing from the longer to the shorter root, if $\theta = \frac{5\pi}{6}$.

Classification of simple Lie algebras

Theorem

The Dynkin diagrams of irreducible root systems are:



On the proof of the theorem

Given any Dynkin diagram of an irreducible root system, one can prove that:

- The Dynkin diagram contains no loops/cycles and is connected (i.e. it's a tree).
- Any node has at most three lines to it.