Proseminar Theoretische Physik

Fusion Rules and the Verlinde Formula

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What is fusion?

fusion = process of taking the short distance product of two fields

goal: find primaries and descendants created by the short distance product of different fields

use: differential equations for fields

Repetition: representation of the Virasoro Algebra

Remember: Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

highest weight representation:

$$L_0|h\rangle = h|h\rangle$$
$$L_{n>0}|h\rangle = 0$$
$$[L_0, L_m] = -mL_m$$

$L_{n>0}$: lowering operator
$L_{-n<0}$: raising operator

Repetition: representation of the Virasoro Algebra

descendant state: $L_{-k_1} \cdots L_{-k_n}|h\rangle \qquad (1 \leq k_1 \leq \cdots \leq k_n)$
is an eigenstate of L_0 with eigenvalue

$$h' = h + k_1 + k_2 + ...k_n = h + N$$

N: level of the state

Verma module V(c,h): subspace generated by $|h\rangle$ and descendants

Hermitian conjugate: $L_n^\dagger = L_{-n}$

inner product of two states $L_{-k_1} \cdots L_{-k_m}|h\rangle$ and $L_{-l_1} \cdots L_{-l_n}|h\rangle$:

$$\langle h|L_{k_m} \cdots L_{k_1} L_{-l_1} \cdots L_{-l_n}|h\rangle$$

Repetition: Virasoro algebra

operator - field correspondance:

$$L_{-n}|h\rangle \qquad \leftrightarrow \qquad \Phi^{(-n)}(w) = \frac{1}{2\pi i}\oint_w dz \frac{1}{(z-w)^{n-1}} T(z)\Phi(w)$$

correlation function including descendant field:

$$\langle \Phi^{(-n)}(w) X \rangle = \mathcal{L}_{-n} \langle \Phi(w) X \rangle \qquad (n \geq 1)$$

$(X = \Phi_1(w_1) \cdots \Phi_N(w_N),$ Φ_i primary fields with conf. weights $h_i)$

$\rightarrow$ reduced to correlator of primaries acted on by differential operator

$$\mathcal{L}_{-n}(w) = \sum_i \left(\frac{(n-1)h_i}{(w_i - w)^n} - \frac{1}{(w_i - w)^{n-1}} \partial_{w_i} \right)$$

Minimal models

- characterized by a Hilbert space made of a finite number of representations of the Virasoro algebra
- describe discrete statistical models (e.g. Ising) at their critical points
- simplicity $\rightarrow$ complete solution

Singular vectors

singular (or null) vector: any state $|\chi\rangle$ - other than the highest weight state - that fulfills $L_n|\chi\rangle = 0$, $(n > 0)$

Singular vectors & their descendants are orthogonal to the whole Verma module V(c,h)!

Quotient out the null submodule of V(c,h)
$\rightarrow$irreducible representation of the Virasoro algebra

Conditions for a state to be singular?

denote basis states as $|i\rangle$

Gram matrix $M_{ij} = \langle i|j\rangle$; $\qquad M^\dagger = M$

block diagonal, each block $M^{(l)}$ corresponds to states of level l

$\rightarrow$ diagonalize $M = U\Lambda U^\dagger$;

$\rightarrow$ for $|a\rangle = \sum_i a_i|i\rangle$: $\quad \langle a|a\rangle = a^\dagger M a = \sum_i \Lambda_i |(Ua)_i|^2$

$\rightarrow$ $\exists$ singular vectors if one of eigenvalues Λ_i vanishes

$\rightarrow$ $V(c,h)$ reducible

Kac determinant

Kac determinant

$$\det M^{(l)} = \alpha_l \prod_{\substack{r,s\geq 1,\\ rs\leq l}} [h - h_{r,s}(c)]^{p(l-rs)}$$

where

$$p(l-rs) = \text{number of partitions of the integer } l-rs$$

$$h_{r,s}(c) = h_0 + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2$$

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}$$

$$h_0 = \frac{1}{24}(c-1)$$

Kac determinant for minimal models

p, p' coprime integers st. $p\alpha_- + p'\alpha_+ = 0$, then:

c and h for minimal models

$$c = 1 - \frac{6(p-p')^2}{pp'}$$

$$h_{r,s} = \frac{(pr - p's)^2 - (p-p')^2}{4pp'}$$

$\Rightarrow$ periodicity: $h_{r,s} = h_{p'-r,p-s}$

$\Rightarrow h_{r,s} + rs = h_{p'+r,p-s}$ and $h_{r,s} + (p'-r)(p-s) = h_{r,2p-s}$

$\Rightarrow$ #(0-vectors) $= \infty \Rightarrow$ finite set of conformal families

$1 \leq r < p'$ and $1 \leq s < p$ (Kac table)

Differential equations for the correlation functions

Suppose $V(c, h_0)$ reducible Verma module with singular vector

$$|c, h_0 + n_0\rangle = \sum_{Y, |Y| = n_0} \alpha_Y L_{-Y} |c, h_0\rangle$$

where

$$\begin{aligned} Y &= \{r_1, \cdots, r_k\} \quad (1 \leq r_1 \leq \cdots \leq r_k) \\ |Y| &= r_1 + \cdots + r_k \\ L_Y &= L_{-r_1} \cdots L_{-r_k} \end{aligned}$$

set corresponding nullfield to zero and insert into correlator

Differential equations for the correlation functions

differential equation for the correlator:

$$\sum_{Y,|Y|=n_0} \alpha_Y \mathcal{L}_{-Y}(z)\langle\Phi_0(z_0)X\rangle = 0$$

where $(X = \Phi_1(w_1)\cdots\Phi_N(w_N), \quad \Phi_i$ primary fields with conf. weights $h_i)$

using

$$\langle\Phi^{(-r_1\cdots,-r_k)}(z_0)X\rangle = \mathcal{L}_{-r_1}(z_0)\cdots\mathcal{L}_{-r_k}(z_0)\langle\Phi(w)X\rangle$$

with

$$\mathcal{L}_{-n}(w) = \sum_{i=1}^{N}\left(\frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i}\right)$$

Example

consider state at level 2

$$\chi = (L_{-2} + aL_{-1}^2)|h\rangle$$

conditions on a and h for χ to be singular:

$$a = -\frac{3}{2(2h+1)}$$

$$h = \frac{1}{16}(5 - c \pm \sqrt{(c-1)(c-25)})$$

differential equation for the correlator:

$$(\mathcal{L}_{-2} - \frac{3}{2(2h+1)}\mathcal{L}_{-1}^2)\langle\Phi(w)X\rangle = 0$$

$$\Rightarrow \left[\sum_{i=1}^{N}\left(\frac{1}{w-w_i}\partial_{w_i} + \frac{h_i}{(w_i-w)^2}\right) - \frac{3}{2(2h+1)}\partial_w^2\right]\langle\Phi(w)X\rangle = 0$$

Example

consider $X = \Phi_1(w_1)\Phi_2(w_2)$

3-point-function:

$\langle \Phi(w)\Phi_1(w_1)\Phi_2(w_2)\rangle = \frac{C_{h,h_1,h_2}}{(w-w_1)^{h+h_1-h_2}(w_1-w_2)^{h_1+h_2-h}(w-w_2)^{h+h_2-h_1}}$

→insert into differential eq.

→ obtain constraints on conformal weights (h, h_1, h_2):

$$h_2 = \frac{1}{6} + \frac{h}{3} + h_1 \pm \frac{2}{3}\sqrt{h^2 + 3hh_1 - \frac{1}{2}h + \frac{3}{2}h_1 + \frac{1}{16}}$$

- choose $h = h_{2,1}; h_1 = h_{r,s} \quad \Rightarrow h_2 \in \{h_{r-1,s}, h_{r+1,s}\}$
- choose $h = h_{1,2}; h_1 = h_{r,s} \quad \Rightarrow h_2 \in \{h_{r,s-1}, h_{r,s+1}\}$

Fusion rules for minimal models

found out:

$$\Phi_{1,2} \times \Phi_{r,s} = \Phi_{r,s-1} + \Phi_{r,s+1}$$
$$\Phi_{2,1} \times \Phi_{r,s} = \Phi_{r-1,s} + \Phi_{r+1,s}$$

Can show:

$$\Phi_{r_1,s_1} \times \Phi_{r_2,s_2} = \sum_{\substack{k=1+|r_1-r_2| \\ k+r_1+r_2=1 mod 2}}^{k=r_1+r_2-1} \sum_{\substack{l=1+|s_1-s_2| \\ l+s_1+s_2=1 mod 2}}^{l=s_1+s_2-1} \Phi_{k,l}$$

The conformal families $[\Phi_{r,s}]$ form a closed set under the operator algebra!

Fusion algebra

OPE

$$\Phi_{h_i}(z)\Phi_{h_j}(w) \sim \sum_h C^{h_k}_{h_i,h_j} \Phi_{h_k}(w)(z-w)^{h_k-h_i-h_j} + \ldots$$

fusion numbers $\mathcal{N}^k_{ij} = \begin{cases} 0, & C^{h_k}_{h_i,h_j} = 0 \\ 1, & \text{otherwise} \end{cases}$

fusion algebra $\Phi_i \times \Phi_j = \sum_k \mathcal{N}^k_{ij} \Phi_k$

Fusion algebra

commutativity: $\mathcal{N}_{ij}^{k} = \mathcal{N}_{ji}^{k}$

associativity: $\sum_l \mathcal{N}_{jk}^{l} \mathcal{N}_{il}^{m} = \sum_l \mathcal{N}_{ij}^{l} \mathcal{N}_{lk}^{m}$

matrix operators N_i: $(N_i)_{j,k} := \mathcal{N}_{ij}^{k}$

$\Rightarrow$ associativity: $N_i N_k = N_k N_i$

properties of the fusion algebra

commutativity $\rightarrow \exists$ matrix S that diagonalizes the N-matrices simultaneously:

$$N_i = S D_i S^{-1}$$

$$\Rightarrow N_{ij}^k = \sum_l \frac{S_{jl} S_{il} (S^{-1})_{lk}}{S_{0l}}$$

How does S look like?

Verlinde:

"The modular transformation $\mathcal{S} : \tau \to -\frac{1}{\tau}$ diagonalizes the fusion rules. "

(Erik Verlinde; Fusion rules and Modular Transformations in 2D Conformal Field Theory)

Verlinde formula

remember: character of a Verma module V(c,h):

$$\chi_{c,h}(\tau) = \text{Tr } q^{L_0 - \frac{c}{24}} \qquad (q := e^{2\pi i \tau})$$

under the action of the modular transformation $\mathcal{S}$ the minimal characters transform among themselves:

$$\chi_{r,s}(-\frac{1}{\tau}) = \sum_{(\rho,\sigma)\in E_{p,p'}} \mathcal{S}_{rs,\rho\sigma}\chi_{\rho,\sigma}(\tau)$$

with $\mathcal{S}_{rs;\rho\sigma} = 2\sqrt{\frac{2}{pp'}}(-1)^{1+s\rho+r\sigma}\sin(\pi\frac{p}{p'}r\rho)\sin(\pi\frac{p'}{p}s\rho)$

not obvious!!

Verlinde formula for minimal models

Verlinde formula for minimal models

$$N_{rs,mn}^{kl} = \sum_{(i,j)\in E_{p,p'}} \frac{\mathcal{S}_{rs,ij}\mathcal{S}_{mn,ij}\mathcal{S}_{ij,kl}}{\mathcal{S}_{11,ij}}$$

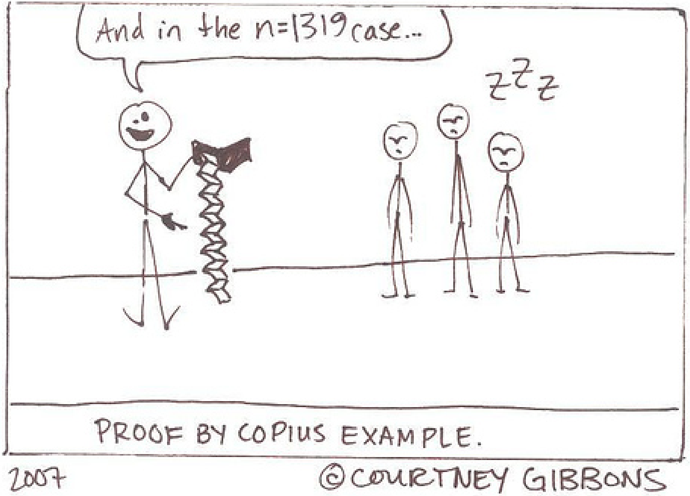
with $\mathcal{S}$ being the matrix of the modular transformation $\mathcal{S}$ in the basis of minimal characters

Verlinde formula

Proof of the Verlinde formula?!

Verlinde:
"In an attempt to convince the reader that our conjecture is correct we will in the next section discuss several examples."

(Erik Verlinde; Fusion rules and Modular Transformations in 2D Conformal Field Theory)


And in the n=1319 case...
zzz
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Example: Ising model

characters of the Ising model:

$$\chi_0(\tau) = \frac{1}{2}\left(\sqrt{\frac{\Theta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\Theta_4(\tau)}{\eta(\tau)}}\right)$$

$$\chi_{\frac{1}{2}}(\tau) = \frac{1}{2}\left(\sqrt{\frac{\Theta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\Theta_4(\tau)}{\eta(\tau)}}\right)$$

$$\chi_{\frac{1}{16}}(\tau) = \frac{1}{\sqrt{2}}\sqrt{\frac{\Theta_2(\tau)}{\eta(\tau)}}$$

modular properties:

$$\Theta_2(-\frac{1}{\tau}) = \sqrt{-i\tau}\Theta_4(\tau)$$

$$\Theta_3(-\frac{1}{\tau}) = \sqrt{-i\tau}\Theta_3(\tau)$$

$$\Theta_4(-\frac{1}{\tau}) = \sqrt{-i\tau}\Theta_2(\tau)$$

$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau)$$