

Fibonacci anyons & Topological Quantum Computers

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Abstract

The following report begins with a brief description of the history of Quantum Computers and an explanation of the concept of Quantum information. Continuing the problem of decoherence is stated and the proposition that topological properties of some 2-D quasiparticles called anyons can help resolve the problem is made.

In the 2nd part of the presentation anyons are explained, both in terms of how they can be physically realized and how the mechanism behind them is theoretically understood. Then the idea of Topological quantum computer is introduced for which anyons are the elementary building blocks.

Further on it is explained how one can compute in the context of an anyonic computational model. All the essential elements required are introduced and a reasonable background for their existence is given.

In the last part a specific anyonic model is studied, namely the Fibonacci anyonic model. It is shown that the property of universality holds for this model and that hence theoretically a Topological quantum computer can be built using these anyons. Finally the field theory that can accommodate such 2-D quasiparticles is identified to be $SU(2)_3$ WZW model coupled to a $U(1)$ gauge field.

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1 Introduction

1.1 History of Quantum Computers

A quantum computer is a computer design which uses the principles of quantum physics to increase the computational power beyond what is attainable by a traditional computer. The idea of a computational device based on quantum mechanics was first explored in the 1970's and early 1980's by physicists and computer scientists such as Charles H. Bennett of the IBM Thomas J. Watson Research Center, Paul A. Benioff of Argonne National Laboratory in Illinois, David Deutsch of Oxford, and the late Richard P. Feynman, Nobel laureate of the California Institute of Technology were pondering the fundamental limits of computation. They understood that if technology continued to abide by Moore's Law, then the continually shrinking size of circuitry packed onto silicon chips would eventually reach a point where individual elements would be no larger than a few atoms. Here a problem arose because at the atomic scale the physical laws that govern the behavior and properties of the circuit are inherently quantum mechanical in nature, not classical.

Quantum computing tends to trace its roots back to a 1959 speech by Richard P. Feynman in which he spoke about the effects of miniaturization, including the idea of exploiting quantum effects to create more powerful computers. (This speech is also generally considered the starting point of nanotechnology.) Of course, before the quantum effects of computing could be realized, scientists and engineers had to more fully develop the technology of traditional computers. This is why, for many years, there was little direct progress, nor even interest, in the idea of making Feynman's suggestions into reality.

In 1985, the idea of "quantum logic gates" was put forth by University of Oxford's David Deutsch, as a means of harnessing the quantum realm inside a computer [4]. In fact, Deutsch's paper on the subject showed that any physical process could be modeled by a quantum computer. Nearly a decade later, in 1994, AT&T's Peter Shor devised an algorithm that could use only 6 qubits to perform some basic factorizations which brought a revolution in the field.

By today, a handful of quantum computers have been built. The first, a 2-qubit quantum computer in 1998, could perform trivial calculations before losing decoherence after a few nanoseconds. In 2000, teams successfully built both a 4-qubit and a 7-qubit quantum computer. Research on the subject is still very active, although some physicists and engineers express concerns over the difficulties involved in up-scaling these experiments to full-scale computing systems. Still, the success of these initial steps do show that the fundamental theory is sound.

1.2 Quantum Information - The Qubit

It is natural to link the study of information and computations to the study of an underlying physical system since information, after all, is something that is encoded in the state of a physical system and a computation is something that can be carried out on an actual physically realizable device. Hence, it was clear already in the early days of quantum theory that classical ideas about information would need revision under the new physics. By understanding that Quantum theory is a much accurate theory in describing the physical world around us than classical theory, it only made sense to also start describing information and computations in a manner that is closer to physical systems.

There exist a number of reasons (phenomena) that aroused by physical systems that led people in start thinking about quantum information theory. Their first clue was the fact that the clicks registered in a detector that monitors a radioactive source are described by a truly random Poisson process while in contrast, there is no place for true randomness in deterministic classical dynamics. Furthermore, in classical information theory there was no room to accommodate the

fact that the act of acquiring information about a physical system should inevitably disturb the state of the system. There is just no counterpart of this limitation in classical physics.¹ Finally, what really led the change from classical information theory to quantum information theory was the discovery from Bell of the phenomenon of entanglement where quantum information can be (in fact, typically is) encoded in non-local correlations between the different parts of a physical system, correlations with no classical counterpart. This property is the one that is essentially being taken advantage of in order to make several quantum algorithms faster than their classical counterparts.

All this gave rise naturally to the idea of qubits, the elementary unit of quantum information analog to the classical bit. Qubits are vectors that live in \mathbb{C}^2 e.g.:

$$0 \rightarrow |0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad 1 \rightarrow |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1)$$

The key difference that makes the qubit advantageous compared to a classical bit and makes the description of information closely related with quantum theory (which is describing the physical systems) is the fact that superpositions of these two states are allowed, i.e. the fact that the qubit lives essentially in a Hilbert space hence the general state of a qubit is given as:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{where} \quad |\alpha|^2 + |\beta|^2 = 1 \quad (2)$$

The true advantages of thinking of information in the above manner though become apparent when one is to consider a combination of two or more qubits. Assume $|\psi_1\rangle \in H_1$, $|\psi_2\rangle \in H_2$ where H_1, H_2 Hilbert spaces. Then a combination of the two:

$$|\psi_{12}\rangle = \sum_{i,j=\{1,2\}} a_{ij} \psi_i \otimes \psi_j \in H_1 \otimes H_2 \quad (3)$$

where a_{ij} can take any value as long as their squares add up to 1, while in the classical case only one a_{ii} was allowed to be equal to 1. For example if we consider the case of two qubits then we see that for qubits there are $2^2 \times 2^2 = 16$ different states the qubit could be in while classically only 2^2 states were allowed. Hence we have an exponential increase of the available state space.

1.3 Quantum Circuits - Representations of Quantum Computers

In devising a model of a quantum computer, we have to generalize the circuit model of classical computation. In doing so, we understand that the logic gates in the classical case become unitary transformations in the quantum case i.e. elements of $SU(2)^2$. This leads naturally to the definition of a quantum circuit: a model for quantum computation in which a computation is a sequence of quantum gates, which are reversible transformations on a quantum mechanical analog of an n-bit register (n-bit classical information input) i.e. on a set of n qubits encoding the information we want to process. A quantum circuit is essentially a representation of a computation. It is also easily understood from the above definition that quantum circuits act on Hilbert spaces (since they act on qubits), and traditionally in a quantum computation is assumed that the initial state of the qubits on which the circuit will act is $|00\dots 00\rangle$.

Classically, there exists a fundamental result in the theory of computation stating that just a few elementary gates suffice to evaluate any function of a finite input. This result means that

¹This observation later led to the no-cloning theorem that states that quantum information can inherently not be copied.

²if we are considering single qubit gates

with very simple hardware components, we can build up arbitrarily complex computations. In the quantum case there exists a similar result stating that if we have in our disposal a set of single qubit rotation gates and any arbitrary two qubit gate³ that can be applied to any two qubits is enough to construct any n-qubit quantum circuit that computes a transformation that comes as close as we please to any element of $U(2^n)$.

1.4 Quantum algorithms

Quantum algorithms are defined as follows: a finite sequence of instructions, or a step-by-step procedure for solving a problem, where each step or instruction can be performed on a quantum computer. Quantum algorithms as mentioned above since they are computations they are represented by quantum circuits. One can talk about the complexity of the algorithm by taking into account the number of resources that are required to be implemented in terms of space need to store information and time it requires to produce an answer given a certain accuracy requirement.

The reason people are interested in quantum computers is because it can be shown that quantum algorithms can produce a result in an amount of time that increases only polynomially with the number of extra qubits as an input compared to a classical algorithm that will produce a result that will increase exponentially in the amount of time with the number of extra bits. This is of course not true for all algorithms but quantum computers are always at least as good as classical ones.

One of the first and simplest quantum algorithms is Deutsch algorithm. In this algorithm we are given a quantum computer known as an oracle that implements the function $f : \{0, 1\} \rightarrow \{0, 1\}$ and we are promised that the function is either constant (0 on both inputs, 0 and 1, or 1 on both inputs) or balanced (returns 1 for half of the input domain and 0 for the other half) the task then is to determine if is constant or balanced by using the oracle. Classically is obvious that the function should be evaluated for both inputs 0 and 1 so that we can basically calculate the value of $f(0) \oplus f(1)$ where \oplus means addition modulo 2. Quantum mechanically though this question is possible to be answered making only a single measurement by using a unitary matrix U_f that depends on the function f that when acts on two qubits it entangles them. The following is the quantum circuit for this algorithm:

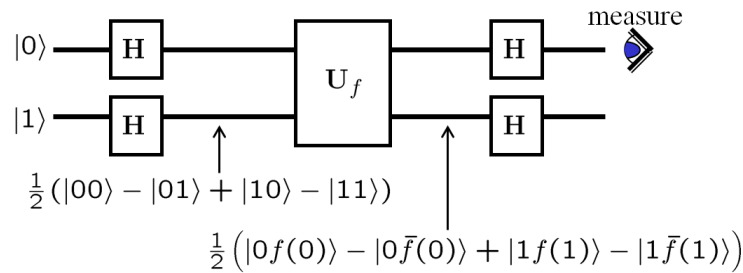


Figure 1: Deutsch algorithm quantum circuit. Although classically the algorithm requires 2 evaluations of the Boolean function f to answer whether it is constant or balanced, the quantum mechanical analogue of the algorithm answers to the question by performing only one measurement taking advantage of the entanglement property.

³usually we refer to a gate called CNOT because it is easier to implement physically but any 2 qubit unitary transformation is equivalent to this gate

where H is the Hadamard gate: $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and U_f is defined as: $U_f := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

(i.e. is equal to the CNOT gate). It can be shown that the final state of the 1st qubit in the algorithm is the following:

$$\frac{1}{2}((1 + (-1)^{f(0) \oplus f(1)}) |0\rangle + (1 - (-1)^{f(0) \oplus f(1)}) |1\rangle) \quad (4)$$

where we can easily see that if we measure the 1st qubit and the outcome is state 0 then this means that $(1 + (-1)^{f(0) \oplus f(1)})$ was not 0 hence $f(0) \oplus f(1) = 0$ and hence we know that the function produces the same outcome for inputs 0 and 1 hence it is constant. Similarly if we measure the 1st qubit and the outcome is state 1 then we know that the function is balanced.

The most well known algorithms are Shor's algorithm for factoring, and Grover's algorithm for searching an unstructured database or an unordered list.

1.5 Decoherence

Decoherence is the loss of coherence or ordering of the phase angles between the components of a system in a quantum superposition. This can be viewed as the loss of information from a system into the environment since every system is loosely locally coupled with the energetic state of its surroundings. As with any coupling, entanglements are generated between the system and environment. These have the effect of sharing quantum information with—or transferring it to—the surroundings. The most important result of the appearance of this phenomenon is the fact that when the system is viewed in isolation, its dynamics are non-unitary any more.⁴ The effects of the environment on the system can be expressed in terms of local operators. To make this effect more concrete, two of the most common errors introduced into the system due to this coupling with the environment, due to the leakage of information are the following:

- Bit flip errors:

$$|0\rangle \rightarrow |1\rangle \quad , \quad |1\rangle \rightarrow |0\rangle$$

- and phase flips:

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \quad , \quad \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

As we've seen in the example of the Deutsch algorithm, the phase angles are very important in order to implement an algorithm. Hence quantum computers are heavily dependent on the correct evolutions of this phases. Therefore one can easily understand that decoherence represents a challenge for the practical realization of quantum computers, since such machines are expected to rely heavily on the undisturbed evolution of quantum coherences. Simply put, they require that coherent states be preserved and that decoherence is managed, in order to actually perform quantum computation.

1.6 Topological properties: A solution to decoherence

It is easy to realize that in order to avoid having the problem of decoherence which is expressed in terms of local operators acting on our data, we have to review the way we encode information

⁴although the combined system plus environment evolves in a unitary fashion

and more specifically we need to encode it in a way that will be independent of the effect of the environment. This immediately implies that information should be stored non-locally.

Topological equivalency is known that is independent of the shape two objects may have. For example the objects below although they look different, topologically they are equivalent, and what determines whether they are equivalent or not is their topological order or their genus (number of holes).

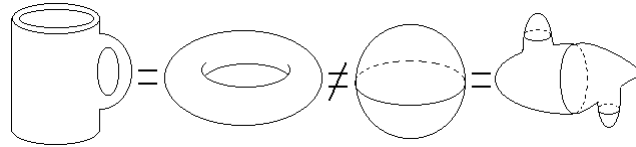


Figure 2: On the left hand side topologically equivalent shapes of genus 1 and on the right hand side topologically equivalent shapes of genus 0. Genus is the number of “holes” a shape has and is what characterizes the topological order of shapes.

The deformation of the shape of the object can be seen as the effect of local operators. However the number of genus can not be attributed to a local operator as it is a non-local property. Therefore combining the above ideas, i.e. our goal and the non-local character of topological degrees of freedom, one can easily conclude that if there was a way to encode information in this non-local degrees of freedom then information would be independent of the environment.

3-dimensional particles, which will comprise the physical system of which our quantum computer is a representation of, do not exhibit topological properties with rich enough structure in order to enable us to take advantage of what we have noted above. Obviously this topological character cannot be sought in the local degrees of freedom of particles like charge etc but a good starting point would be the symmetries of 3d particles under their exchanges, i.e. their statistical symmetries. Exchange statistics describe the change in the wave function of two identical particles, when they are exchanged. Its properties need to be compatible with the symmetry imposed by indistinguishability. As an important consequence these changes are independent of many details of the system. Consider, for example, the case where the exchange is not a mathematical procedure, but a physical process of moving two particles along an exchange path. The effect of this transport on the wave function should not depend on the particular shape of the path taken by the particles when they are exchanged or the speed the path is traversed. Nevertheless, the evolution might still depend on some global, topological characteristics of the path, such as the number of times the particles are exchanged. Statistical evolutions are hence topological in their nature.

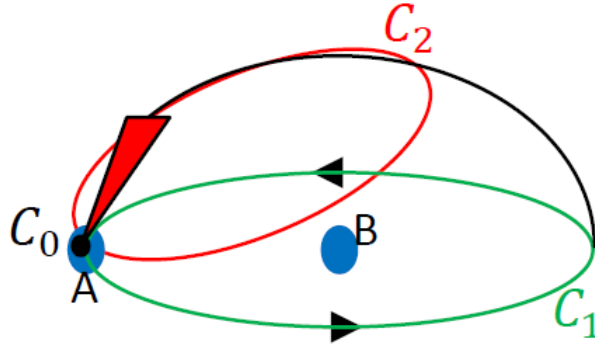


Figure 3: The topological nature of statistical evolutions. It can be easily understood that, due to the fact that in 3d we can continuously deform the path C_1 to C_2 and essentially to C_0 which corresponds to the point path i.e. to the case where no path was traversed at all, the process of winding a particle around the other should have a trivial effect on the total wavefunction.

Due to indistinguishability property, a common wave function, e.g. $|\psi_{AB}\rangle$, is adopted to describe a system of two particles. As we can see in Figure 3 above, after a winding of these particles the path traversed, is topologically equivalent to the point hence the effect of the winding should be trivial. Hence the wavefunction after the winding remains $|\psi_{AB}\rangle$ i.e. $U^2 |\psi_{AB}\rangle = |\psi_{AB}\rangle$ where $U = e^{i\theta}$ corresponds to the unitary effect of the exchange of particles A and B on the common wavefunction. Therefore since $U^2 = I$ there exists two solutions of the equation and hence two particles corresponding to the two different ways of behaving under the exchanges. The two solutions are $U = e^{i(2\pi+2\pi n)}$ and $U = e^{i(\pi+2\pi n)}$ where n is an integer. This is the reason that 3d particles split into Bosons and Fermions.

The above discussion leads us to the conclusion that no particles can accommodate interesting topological properties and the reason is their 3d nature. But what if we consider quasiparticles? It is known that these can exhibit properties completely different from their constituent particles. They could even live effectively in 2 dimensions rather than 3. Would this property make them interesting?

2 Anyons-The bricks for building a topological quantum computer

2.1 “Any”ons and their physical realization

Statistics is spectacularly manifested in two-dimensional systems. Continuing the study regarding the topological properties under exchange of particles from the previous chapter but with the particles living in 2d now one can easily see that the picture we had before does not hold any more. More specifically the equivalence of a path formed due to the winding of particles and the point are not topologically equivalent any more as one can see in figure 4:

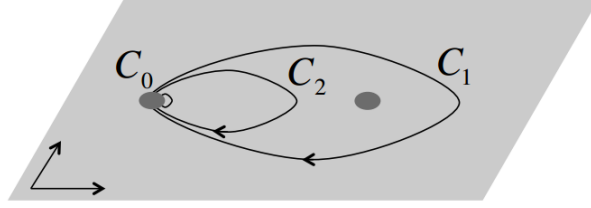


Figure 4: In 2D contraction of the traversed path, after winding of one particle around the other, to the point circuit is not possible any more. Hence the effect of the winding does not have to be trivial.

Removing the 3rd dimension removes the ability to continuously deforming the path to the point. Therefore the requirement that $U^2 = I$ is lifted. Immediately one observes then that the phase θ of the unitary transformation U can take any value. This is also what gives the name to these quasiparticles, i.e. “any”-ons. More specifically these kind of quasiparticles are called Abelian anyons for reasons that will become apparent soon.

Exotic wave functions of particles can emerge that give rise to these 2d quasiparticles called “anyons”. To understand their appearance consider the case where a 3d system is subject to the following spatial potential:

$$V(r) = V_{xy}(x, y) + V_z(z) \quad (5)$$

This potential after employing the method of separation of variables can give rise to the following wavefunction:

$$\Psi(r) = \Psi_{xy}(x, y)\Psi_z(z) \quad (6)$$

where we see that essentially the system in XY plane is 2d and its dynamics are independent of the 3rd dimension.

Apart from Abelian anyons there also exist Non-abelian anyons. The property that gives these characterizations to anyons is the dimension of the Hilbert space of the 2d wavefunction that describes these anyons. Abelian anyons live in a 1d Hilbert space hence a unitary transformation applied on this space can only be of the form mentioned above, i.e. $U = e^{i\theta}$ an abelian phase element of $U(1)$. However if the wavefunction of the anyon lives in a higher dimensional Hilbert space then the unitary transformation U mentioned before is an element of $SU(N)$ where N is the dimension of the Hilbert space. Hence particles that are described by these type of wavefunctions are called non-abelian. One cannot help but to notice that these unitary transformations are basically equivalent to quantum gates acting on qubits the elements of a quantum circuit. This property of non-abelian anyons is exactly what made people start thinking about how they could use them to simulate a quantum circuit and hence a quantum computer.

The important thing to consider now is whether these quasiparticles, the non-abelian anyons, are physically realizable. There exist a number of requirements that need to be satisfied to be able to physically realize these particles. First of all by definition their wavefunction must live in a degenerate state space. Second this degenerate state space should be the ground state and there should exist a finite energy gap ΔE between this state and the first excited energy state. The reason for this requirements is to preserve the 2dimensionality of the wavefunction. If the above are not satisfied then is possible for the state to jump to another energy level (higher or lower) and then the dynamics of the evolution of the wavefunction would also depend on the 3rd dimension z . Third, the exchanges of the quasiparticles should be done in an adiabatic way

so that the transformations applied on the anyons will be well controlled. If the exchange is performed too quickly then two disastrous effects may take place. Firstly we may introduce enough energy in the system so that it will be excited in a higher energy level, and secondly we may offer enough energy in the system so that during the exchange pairs of particles antiparticles may appear and hence will result in the evolution of the anyons not being unitary or not being the one that was expected to be. Fourth requirement for physically realizing non-abelian anyons is the fact that non-abelian anyons should be far apart so that there is no overlap between their wavefunctions and hence no spontaneous change of the type of non-abelian anyon can be observed. This requirement essentially guarantees that the only process that can cause a change of the type of the anyon is the process of braiding, i.e. the process of exchanging particles. The final requirement stems from the fact that we want these non-abelian anyons to be independent of the environment and it states that all local operators should have vanishing correlation functions apart from the identity so that the environment which is represented in terms of local operators can not cause a change of the type of anyon.

It was soon realized that anyons can be encountered in physical systems with effective two-dimensional behavior. For example, gases of electrons confined on thin films in the presence of sufficiently strong magnetic field and at a sufficiently low temperature give rise to the fractional quantum Hall effect. In this effect is observed that groups of electrons forming quasiparticles appear with conductance given in terms of a fractional number ν . This implies that these quasiparticles have fractional charge. The excitations for $\nu = \frac{1}{3}$ have been accurately described by Laughlin and it was shown that they corresponded to abelian anyons at low energies. For $\nu = \frac{5}{2}$ the excitations are described by the excitations of the Moore-Read states and are believed to correspond to non-abelian anyons and more specifically to Ising anyons. The latter have not been observed yet while the abelian ones have indeed been observed. It has not been proved that indeed they obey fractional statistics i.e. anyonic statistics but the fractional charge has been confirmed and is expected that fractional statistics will be observed too. It is important to also note that these states have been shown to emerge in the context of a conformal field theory and more specifically in the context of the WZW models. Beyond the fractional quantum Hall effect other two dimensional systems have emerged which theoretically support anyons. These range from superconductors and topological insulators to spin lattice models.

2.2 Origin of unitary transformations under anyon exchanges - The geometric phase

After having understood how the non-abelian anyons can be physically realized we also have to investigate the mechanism that gives rise to the unitary transformations being applied on the degenerate state space of non abelian anyons when two non abelian anyons are exchanged. To understand this mechanism we first have to study another phenomenon called Berry's geometric phase [6].

This geometric phase is a phase acquired over the course of a cycle, when a system is subjected to cyclic adiabatic processes, which results from the geometrical properties of the parameter space of the Hamiltonian. The phenomenon was first discovered in 1956 and generalized in 1984 by Berry. The simplest example where this phenomenon is observed is the Aharonov-Bohm effect. So to make the concept of the geometric phase more concrete consider the following scenario. Assume that we have a parametric space M that parametrizes the degrees of freedom related to the coupling of a system we have in our disposal to its environment. Furthermore, assume that we parallel transport the state space of our system along a circuit on this parametric space as shown on figure 5:

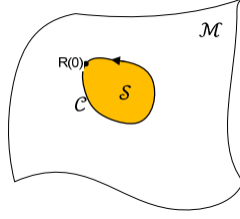


Figure 5: Parametric space \mathcal{M} , that parametrizes the degrees of freedom related to the coupling of the system to the environment, on which the state space of our system is parallel transported along a circuit C .

Contrary to what one would expect, a state in this state space will not acquire just a dynamic term after the state space returns to its initial position, but it will also acquire a geometric term. The state after the parallel transporting is given by the following formula:

$$|\psi(t)\rangle = e^{i\gamma_n(C)} e^{-\frac{i2\pi}{\hbar} \int_0^T dt E_n(R(t))} |\psi(0)\rangle \quad (7)$$

where $E_n(R(t))$ is the energy of the state n at point $R(t)$ where $R(t)$ is the point on the parametric space at time t and

$$\gamma_n(C) = i \oint_C \langle n, R(t) | \nabla_R n, R(t) \rangle dR = \oint_C A_\mu dR^\mu = \oint_S \frac{1}{2} F_{\mu\nu} dR^\mu \wedge dR^\nu \quad (8)$$

where $|n, R(t)\rangle$ is the state at energy level n and point $R(t)$, S is the surface related to the circuit traversed on the parametric space, A_μ is the vector potential related to the parametric space and $F_{\mu\nu}$ the corresponding tensor field. This tensor field essentially characterizes the curvature of the parametric space. The vector potential gauge transformations are given as following:

$$A_\mu \rightarrow A_\mu - \partial_\mu a_n \quad (9)$$

and hence $F_{\mu\nu}$ which is defined in terms of the vector potential as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (10)$$

can be easily understood to be gauge invariant. If this is not also diffeomorphic invariant this means that $F_{\mu\nu} \neq 0$ which implies that the curvature of the parametric space is non-trivial. Then there exist two cases. First we might parallel transport a non-degenerate state space. To see what happens consider the example of the sphere as in figure 6:

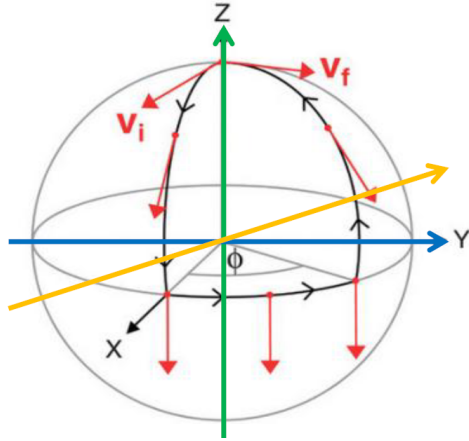


Figure 6: Parallel transporting an initial state V_i along a circuit on a sphere results in a state V_f that is at an angle θ relative to the initial state. Hence one concludes that the effect of this parallel transporting is a transformation element of the group $U(1)$.

where we see that parallel transporting an initial state V_i along the circuit shown results in a final state V_f which is at an angle relative to the initial state. This means that the transformation of the state is an element of the $U(1)$ group, in which case the transformations are called abelian geometric phase evolutions.

Secondly we might parallel transport a degenerate state space along a circuit. Again consider the example of the sphere to see the result of this action like in figure 7:

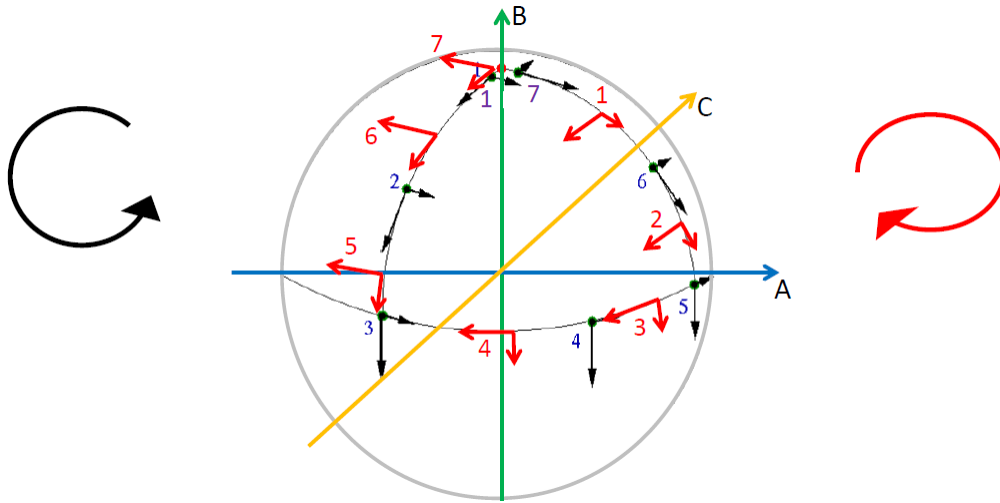


Figure 7: Parallel transporting a 2 dimensional degenerate state space on a sphere results in a transformation being applied on it which is an element of $SU(2)$. In the example given here the transformation will be again that both vectors are at an angle θ compared to the initial ones. Also is easy to notice that parallel transporting clockwise or counter-clockwise on the circuit results in two different transformations, and more specifically in two transformations that are the inverse of each other.

where we see that when for example parallel transporting a degenerate state space of degeneracy 2 results in a transformation of the state space that is an element of $SU(2)$ ⁵. Further note that parallel transporting clockwise or anticlockwise results in a different transformation, where for the specific example one is the inverse transformation of the other. This also holds for the non degenerate state space as well. In general one can show that the transformation of a degenerate state space of degeneracy N after such a parallel transporting will be an element of $SU(N)$ and hence transformations in this case are called non-abelian geometric phase evolutions. Which element will be depends on the circuit that will be traversed and hence on the geometry of the parametric space.

One can define a map $\Gamma_A(C)$ called holonomy as: $\Gamma_A(C) = e^{i\gamma_n(C)}$, i.e. in terms of the geometric phase. This is basically a map from the space of circuits on the parametric space to the space of operators being applied on the state space of the system to be parallel transported. One can show that the following properties hold for this map [1]:

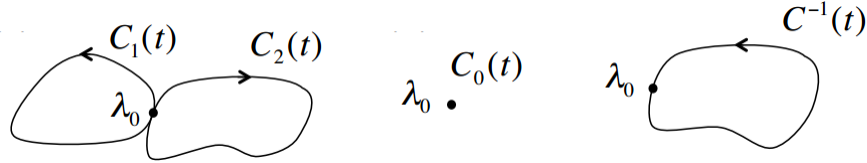


Figure 8: From left to right: The composite loop that spans C_1 and C_2 , the point circuit and finally the counterclockwise traversed circuit

1. $\Gamma_A(C_1 * C_2) = \Gamma_A(C_1) \cdot \Gamma_A(C_2)$ i.e. that the map is a homomorphism.
2. $\Gamma_A(C_0) = I$ where C_0 is the point circuit which means that traversing the trivial path results in no change due to geometry in the state space (this is how fermions and bosons are retrieved since the topology there shows the equivalency of the path to the point)
3. $\Gamma_A(C^{-1}) = \Gamma_A^{-1}(C)$ i.e. that when the path is traversed in the opposite manner (C^{-1} being traversed clockwise or anticlockwise if the path C is defined to being traversed anticlockwise or clockwise respectively) then the inverse operator is being applied on the state space
4. $\Gamma_A(C \cdot f) = \Gamma_A(C)$ where f is a function of time hence this implies that the operator that will be applied on the state space does not depend on how fast the circuit is traversed. This property encodes the requirement of adiabaticity we mentioned before.

One can easily see that the first 3 properties are enough to show that the elements generated by this map are elements of a group.

Now to return to our original goal i.e. to manage to describe the mechanism that gives rise to the unitary transformations being applied on the non abelian anyons after their exchanges we have to further make the following assumptions:

1. First assume that there exist a complicated Hamiltonian that can relate the degrees of freedom of the parametric space to anyons coordinates, i.e. to relate the circuit traversed in the parametric space to the path traversed by the non-abelian anyons
2. Second assume that the tensor field $F_{\mu\nu}$ is confined to the position where the anyon resides. This way we make sure that the unitary transformations being applied on the anyons do not depend on the geometry of the system but rather on its topology.

⁵In the specific example the transformation would be of the form $e^{i\theta} I_{2 \times 2}$

Given the above assumptions hold one can see that non-abelian geometric phase evolutions can be mapped to non-abelian anyons unitary transformations.

2.3 The idea of a Topological Quantum computer

Following the above considerations one realizes that if there was a way to keep track of these unitary transformations then it could be possible to simulate a quantum circuit. The idea to implement this gave birth to topological quantum computers, which is a system where quasi-particle's worldlines are used in order to keep track of the unitary transformations being applied on the Hilbert space of the anyons.

The first result towards showing the equivalence of an anyonic system and a quantum circuit came in 1997 when Kitaev showed that is possible to simulate a quantum circuit using anyons. In 2000 Kitaev, Freedman and Wang showed that the reverse is also true and hence the equivalency holded. However this is not enough to implement a Topological quantum computer as it is also required by a computational model to exhibit the property of universality to be characterized as a quantum computer. In order to show that this is true for a Topological quantum computational model, i.e. that is true for an anyonic model we need to first understand how one can compute in the context of such a model. The study of how to do this will be the subject of the next chapter.

3 Anyonic computational models

As always to study a system (in this case the system where we exchange anyons) we need to identify it's group of symmetries, so first of all we need to examine if such a group exists. Our intuition should be that such a group must exist since in the previous chapter we showed the equivalence of non-abelian geometric phase evolutions to non-abelian anyons unitary transformations and after this we showed that there was indeed a group related to the non abelian geometric phase evolutions. Hence is only natural that such a group will exist for non-abelian unitary transformations as well.

3.1 Braid group

A first step towards showing that the group exists is to identify its elements, i.e. we need a set with distinct elements. This set is identified if we make the following observation. The quasiparticle worldlines cannot cross, hence braiding of two anyons belong in distinct topological classes as you can see in the following figure:

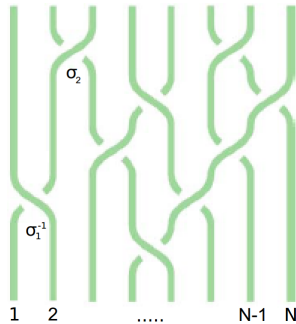


Figure 9: Quasiparticle worldlines and the elements of the Braid group. Notice the existence of the inverse Braid.

Therefore is obvious that we can identify the elements of a set which we shall call the Braid set to be these distinct topological classes which basically are defined as:

$$\sigma_i := \text{the exchange of the } i^{\text{th}} \text{ and } (i + 1)^{\text{th}} \text{ anyons}$$

To see that the elements of this set form indeed a group we have to further observe that the following properties hold:

1. The combination of several of elementary two particle braids can lead to any arbitrary long braid. Therefore we have closure property.
2. The inverse of any braid exists
3. The vacuum is part of the set, i.e. the braid when nothing happens.
4. The braidings are associative.

Hence this set is indeed a group and is called the Braid group. The representations of this group $\rho(\sigma_i)$ are essentially the unitary transformations that are applied on the anyons when they are exchanged.

There also exists two other conditions that have to be satisfied by the elements of a braid group and these are:

1. that exchanges of disjoint particles commute:

$$\sigma_k \sigma_j = \sigma_j \sigma_k \quad |j - k| \geq 2 \tag{11}$$

2. the Yang-Baxter relation:

$$\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1} \quad j = 1, 2, \dots, n - 2 \tag{12}$$

3.2 Fusion

One other process that can take place when one considers two anyons is Fusion. This is the effect of bringing two particles together. In particular one can think of this process as the equivalent of taking two anyons, bringing them together and placing them into a box and then trying to read the distribution that corresponds to the particle that is in the box.

There exists a set of rules relating the particles that are fused to their outcomes. These rules are encoded in an algebra, the fusion algebra:

$$a \times b = \sum_c N_{ab}^c c \tag{13}$$

where N_{ab}^c can be matrices. Therefore the elements of this algebra are also the anyons that are allowed to exist in a given anyonic computational theory. The list with the allowed particles forms a finite set and the particles in this set are characterized by their charge. This charge of anyons is conserved under local operations as it was mentioned in the first part of the report.

The coefficients N_{ab}^c are very important and are the ones who determine the algebra. They also determine whether the anyons fused are abelian, in which case $N_{ab}^c = 1$ or non-abelian, in which case $\sum_c N_{ab}^c > 1$. The fusion algebra also satisfies the associativity property:

$$\sum_e N_{ab}^e N_{de}^c = \sum_e N_{bd}^e N_{ea}^c \quad (14)$$

and the commutativity property:

$$N_{ab}^c = N_{ba}^c \quad (15)$$

This algebra can be shown to be equivalent to the algebra relating minimal models:

$$\phi_i \times \phi_j = \sum_k N_{ij}^k \phi_k \quad (16)$$

since it can be shown that this algebra also satisfies associativity and commutativity, therefore it is possible to map minimal models to anyonic models.

3.2.1 Fusion space

One important concept that is necessary in order to study anyonic computational models is the concept of Fusion spaces. These are Hilbert spaces subspaces of the space of all possible fusion outcomes. For these fusion spaces the fusion algebra takes the following form:

$$V_a \otimes V_b = \oplus_c N_{ab}^c V_{ab}^c \quad (17)$$

where V_a, V_b are complex irreducible representations of anyons a and b, and V_{ab}^c is the fusion space. These fusion spaces are important because the logical states $|0\rangle$ and $|1\rangle$ in an anyonic computational model will be encoded on one of these fusion spaces. For fusion spaces the following relation that relates the quantum dimension of anyons a and b and the dimensionality of the fusion space is true:

$$d_a d_b = \sum_c N_{ab}^c d_c \quad (18)$$

where d_i $i = \{a, b, c\}$ are the quantum dimensions of the anyons to be fused and the dimensionality of the fusion space where the outcome anyon lives. The simplest non-abelian fusion example where one can see that this formula indeed holds is the fusion of two spin- $\frac{1}{2}$ particles:

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \quad (19)$$

where indeed the dimensions match i.e. $2 \cdot 2 = 1 + 3$.

We can also identify a basis for the Hilbert space of the fusion outcome. To find possible basis consider the example of Fibonacci anyons fusion. In an anyonic computational model with Fibonacci anyons, the only non-trivial fusion rule is the following:

$$\tau \otimes \tau = 1 \oplus \tau \quad (20)$$

where 1 represents the vacuum and τ is the Fibonacci anyon. Fusions can in general be represented by fusion trees. Assume that we have in our disposal only 3 Fibonacci anyons and that we start fusing them. Further assume that at the end we measure the charge of the produced anyon and we find this to be equal to τ . The whole process can be split into two cases that are represented by fusion trees as follows:

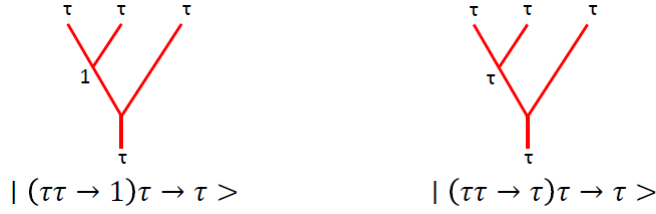


Figure 10: Fusion trees represent orthogonal basis elements of the fusion Hilbert space

It can be easily understood that since particles with different charges are distinct then these two fusion trees are also distinct. Therefore these fusion trees are representing orthogonal basis elements of a Hilbert space (since the anyons live in Hilbert spaces) and these are the only elements of the basis hence the Hilbert space with fixed initial state of 3 τ and outcome 1 τ has dimension 2. Have we had one more possibility as the outcome of the fusion then the Hilbert space would have been 3-dimensional. Continuing down this road one can see that the dimension of the Hilbert space depends on the number of the in-between states.

3.3 The F,R and B matrices

F matrix Two very essential tools necessary to study any anyonic computational model are the F and R matrices. The F matrix is a matrix that results from the fact that fusion is associative. The consequence of this fact can be represented in terms of fusion trees as following:

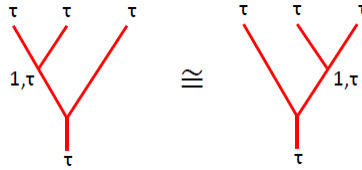


Figure 11: Order of the fusion is irrelevant

which means that the order of the fusion is irrelevant. As it was already mentioned these fusion trees correspond to orthogonal basis of a Hilbert space, hence changing the order of the fusion amounts to just changing the basis elements. This changing of the basis elements is a unitary transformation which can be represented by a matrix, the F matrix:

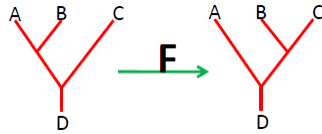


Figure 12: Application of the F-matrix

R matrix There also exist the R matrix. This is a matrix that encodes the effect of exchanging two anyons a and b before fusing them. If we consider the case of a single outcome c then it is easy to see that the effect of this is just a phase being gained by the new anyon c. So in this case R is not really a matrix but rather just a phase as one can observe in the following figure:

$$R_{a,b} \begin{array}{c} a \quad b \\ \diagdown \quad / \\ \quad \quad c \end{array} = \begin{array}{c} b \quad a \\ / \quad \diagdown \\ \quad \quad c \end{array} = R_c^{b,a} \begin{array}{c} b \quad a \\ \diagdown \quad / \\ \quad \quad c \end{array}$$

Figure 13: The R-phase

In the case that there are several anyons fusing and hence there are several in-between states between the two anyons a and b exchanged and the final fusion outcome c then R takes the form of a diagonal matrix:

$$\begin{array}{c} a \quad b \\ \diagdown \quad / \\ \mu \\ \downarrow \\ c \end{array} = \sum_{\mu'} (R_{ba}^c)^{\mu'}_{\mu} \begin{array}{c} a \quad b \\ / \quad \diagdown \\ \mu' \\ \downarrow \\ c \end{array}$$

Figure 14: The R-matrix

B matrix There is one more complicated scenario to consider and that is the case when the two anyons a and b that are going to be exchanged before fused do not have a direct fusion outcome. To counterattack this problem we have to apply an F matrix before encoding the effect in an R matrix so that we will rearrange the basis to a basis where the two anyons do indeed have a direct fusion outcome, i.e. we encode the effect in a new matrix called B matrix given as following:

$$B = FRF^{-1} \tag{21}$$

This is shown diagrammatically for the example of exchanging two Fibonacci anyons that do not have a direct fusion outcome in figure:

$$\begin{array}{c} \tau \quad \tau \\ | \quad | \\ a \quad b \quad c \end{array} \xrightarrow{F} \begin{array}{c} \tau \quad \tau \\ \diagdown \quad / \\ \hat{b} \\ \downarrow \\ c \end{array} \xrightarrow{R} \begin{array}{c} \tau \quad \tau \\ / \quad \diagdown \\ \hat{b} \\ \downarrow \\ c \end{array} \xrightarrow{F} \begin{array}{c} \tau \quad \tau \\ | \quad | \\ a \quad \hat{b} \quad c \end{array}$$

Figure 15: F-matrix results in a change of basis from: $\begin{pmatrix} |1\tau1\rangle \\ |1\tau\tau\rangle \\ |\tau1\tau\rangle \\ |\tau\tau1\rangle \\ |\tau\tau\tau\rangle \end{pmatrix}$ **to** $\begin{pmatrix} |111\rangle \\ |1\tau\tau\rangle \\ |\tau1\tau\rangle \\ |\tau\tau1\rangle \\ |\tau\tau\tau\rangle \end{pmatrix}$

so that the anyons to be exchanged will have a direct fusion outcome

These B matrices essentially are the representations of the Braid group elements, the unitary transformations that were mentioned in the definition of non-abelian anyons. These encode all the information we need to know about how to compute in an anyonic model and hence one concludes that F and R matrices are all we need to know to determine computations in an anyonic computational model.

3.3.1 The Pentagon and Hexagon equations

In order to determine these F and R matrices we need to impose some constraints on them and hence we need to find equations that will provide us with these constraints. The solution again comes from the associativity property of fusion.

Pentagon equation First by considering the equivalence of the following due to associativity:



Figure 16: The two tree diagrams are equivalent and hence there must be a path formed by consecutive applications of F matrices that transforms from one diagram to the other

one can see that there are two paths constructed by applications of F matrices that can be followed in order to transform from one to the other diagram:

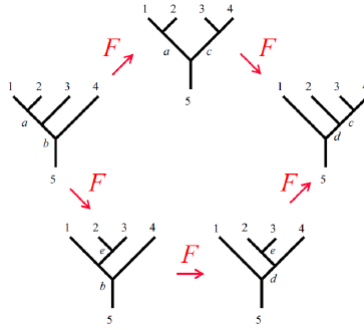


Figure 17: The pentagon equation

and this results to the pentagon equation:

$$(F_{12c}^5)_a^d (F_{a34}^5)_b^c = \sum_e (F_{234}^d)_e^c (F_{1e4}^5)_b^d (F_{123}^b)_a^e \quad (22)$$

Cases of fusion of more than 4 anyons can be reduced down to this hence no more constraints can be found by considering more anyons.

Hexagon equation This equation appears if we further allow exchanges of anyons in trying to reach equivalency of two diagrams. The hexagon equation in diagrammatic form looks as following:

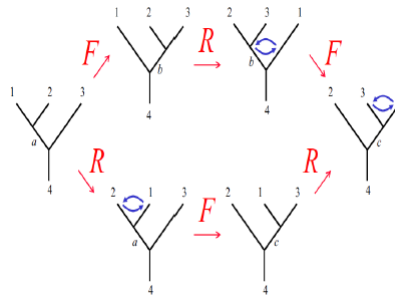


Figure 18: The Hexagon equation

and has the following mathematical form:

$$\sum_b (F_{231}^4)_b^c R_{1b}^4 (F_{123}^4)_a^b = R_{13}^c (F_{213}^4)_a^c R_{12}^a \quad (23)$$

Again cases where more anyons take part in the process can be reduced down to this elementary case and hence again no more conditions can be obtained if we have more anyons.

Pentagon and Hexagon equations provide therefore all the constraints we can impose on the F and R matrices and hence they completely determine them.

3.3.2 Relation to S-matrix of Verlinde formula

The S-matrix is used in diagonalizing the coefficients of the fusion algebra of minimal models. It appears in the so called Verlinde formula. This formula has an analogue in the anyonic world as well since as we already saw it is possible to map anyonic models to minimal models. The Verlinde formula takes the following form:

$$(N_a)_b^c = \sum_d S_b^d \left(\frac{S_a^d}{S_1^d} \right) (S^{-1})_d^c \quad (24)$$

where the S matrices are to be translated as:

$$S_a^b = \frac{1}{\mathcal{D}} \begin{array}{c} \text{Diagram showing two pairs of particles (a and b) appearing, then exchanging, and then annihilating again. The diagram consists of two diamond shapes. The left diamond is red and labeled 'a', and the right diamond is blue and labeled 'b'. Arrows indicate the flow of particles: two red arrows enter from the left, two red arrows exit to the right, two blue arrows enter from the left, and two blue arrows exit to the right. The two diamonds are connected by arrows, indicating an exchange of particles between them. The entire process is enclosed in a larger diamond shape, representing the S-matrix element.$$

Figure 19: S-matrix encodes the following process: two pairs of particle-antiparticle of types a and b appearing, then the particles exchanging and then annihilating again.

which physically describes the situation where two pairs of particles-antiparticles of species a and b are created, then the particles a and b are exchanged and then the particles-antiparticles are fused again. Again the S-matrices diagonalize the coefficients of the fusion algebra and they basically encode all the available information for the anyonic system. Hence the information encoded by the S-matrices is equivalent to the information encoded by the F and R matrices we saw in the previous subsection.

4 The Fibonacci anyonic model

The Fibonacci anyonic model is one of the most interesting anyonic computational and at the same time one of the simplest. The study of this model was the reason of the background material covered in the previous sections.

In this model as was mentioned before the only non-trivial fusion rule is the one given by equation (20). The first step in studying this model is to determine how we want to encode the logical states with the criterion that the encoding will allow us to achieve universality. As was mentioned before these states will be encoded on a fusion Hilbert space. The naive way to do this would be the following:

$$\begin{aligned}
|0\rangle &= |(\bullet, \bullet)_1\rangle = \text{diagram with two dots in an oval labeled } \mathbf{1} = \text{diagram with two lines crossing labeled } \mathbf{1} \\
|1\rangle &= |(\bullet, \bullet)_\tau\rangle = \text{diagram with two dots in an oval labeled } \tau = \text{diagram with two lines crossing labeled } \tau
\end{aligned}$$

Figure 20: The naive way of encoding the logical states in the Fibonacci anyonic computational model

However one can easily see that the fusion outcomes, the charges related to the logical states are different. As was mentioned before charges of anyons are conserved under local operators. The single qubit gates that we want to have in our computational model are also local operators, i.e. unitary transformations. But a unitary transformation will not be able to change the charge and hence will not be able to rotate state $|0\rangle$ to state $|1\rangle$ hence we will not be able to span $SU(2)$. So this is the wrong way to encode the logical states. Instead we can avoid this problem if we encode them in the following way:

$$\begin{aligned}
|0\rangle &= |((\bullet, \bullet)_1, \bullet)_\tau\rangle = \text{diagram with two dots in an oval labeled } \mathbf{1}, \text{ and a third dot outside labeled } \tau = \text{diagram with three lines crossing labeled } \mathbf{1} \\
|1\rangle &= |((\bullet, \bullet)_\tau, \bullet)_\tau\rangle = \text{diagram with two dots in an oval labeled } \tau, \text{ and a third dot outside labeled } \tau = \text{diagram with three lines crossing labeled } \tau \\
|N\rangle &= |((\bullet, \bullet)_\tau, \bullet)_1\rangle = \text{diagram with two dots in an oval labeled } \tau, \text{ and a third dot outside labeled } \mathbf{1} = \text{diagram with three lines crossing labeled } \mathbf{1}
\end{aligned}$$

Figure 21: The encoded logical qubits

where we can see that the problem is lifted. Also the appearance of the 3rd state is not to worry us as again the charge for this state is different than for the other two so no unitary transformation can introduce the 3rd state in a calculation unless we had this state right from the beginning in our initial data.

The next step is to determine the unitary transformations that are being applied on the anyons when they are exchanged, and to do this we need to determine the F and R matrices of the system. The trivial F matrices are the following as one can verify easily:

$$F_1^{\tau\tau\tau} = F_\tau^{1\tau\tau} = F_\tau^{\tau 1\tau} = F_\tau^{\tau\tau 1} = 1 \quad (25)$$

and by plugging these into the pentagon equation one can obtain the only non-trivial F -matrix [3]:

$$F_\tau^{\tau\tau\tau} = \begin{pmatrix} \frac{1}{\varphi} & \frac{1}{\sqrt{\varphi}} \\ \frac{1}{\sqrt{\varphi}} & -\frac{1}{\varphi} \end{pmatrix} \quad (26)$$

where $\varphi = \frac{(1+\sqrt{5})}{2}$ is the golden ratio.

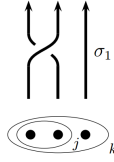
For the R matrix, we again find the trivial ones by directly observing this [3]:

$$R_\tau^{1\tau} = R_\tau^{\tau 1} = 1 \quad (27)$$

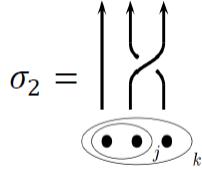
and by plugging these and the F matrices we found above in the Hexagon equation we obtain the only non-trivial R matrix which is:

$$R^{\tau\tau} = \begin{pmatrix} e^{i\frac{4\pi}{5}} & 0 \\ 0 & -e^{i\frac{2\pi}{5}} \end{pmatrix} \quad (28)$$

These two matrices then will give us the unitary transformations that are employed after the exchange of two anyons, that we are essentially looking for. These are given as $U_i = \rho(\sigma_i) = B_i = (FRF^{-1})_i$ and there exists two of them corresponding to the two following Braid group elements:



and



For these braids then the unitary transformations are [3]:

1.

$$\rho(\sigma_1) = \begin{pmatrix} e^{-i\frac{4\pi}{5}} & 0 & 0 \\ 0 & -e^{-i\frac{2\pi}{5}} & 0 \\ 0 & 0 & -e^{-i\frac{2\pi}{5}} \end{pmatrix}$$

2.

$$\rho(\sigma_2) = \begin{pmatrix} \frac{-e^{-i\frac{\pi}{5}}}{\varphi} & -i\frac{e^{-i\frac{\pi}{10}}}{\sqrt{\varphi}} & 0 \\ -i\frac{e^{-i\frac{\pi}{10}}}{\sqrt{\varphi}} & -\frac{1}{\varphi} & 0 \\ 0 & 0 & -e^{-i\frac{2\pi}{5}} \end{pmatrix}$$

which are acting on a vector of the form: $\begin{pmatrix} |0\rangle \\ |1\rangle \\ |N\rangle \end{pmatrix}$ and hence one can easily see that

indeed the final state $|N\rangle$ can not be introduced in the system by braiding processes.

The origin of the name of the model is understood once someone considers the dimension of the Hilbert space on which the initial data are encoded. So if we start fusing Fibonacci anyons (that represent our initial data) with the only constraint that no two consecutive vacuum particles can be part of this process then we get the following results. For fusing 2 Fibonacci anyons the dimension of the space is 2 but as it was explained before this will have no meaning in our theory because it cannot be translated to logical qubits. For fusing 3 Fibonacci anyons, as we saw above the dimension of the fusion Hilbert space is 3 and corresponds to the 3 states we saw. The following diagram represents what happens as we keep fusing Fibonacci anyons:

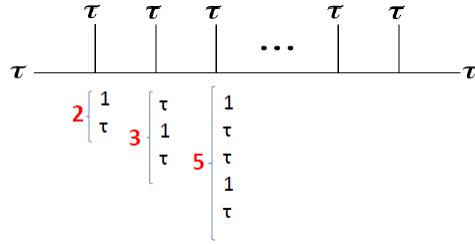


Figure 22: If we start fusing Fibonacci anyons the dimension of the Fusion Hilbert space increases with the number of anyons fused as the Fibonacci sequence

It is understood that the dimension of the Fusion Hilbert space increases as the Fibonacci sequence as we keep fusing more and more anyons at the beginning. This can be more clearly seen in a Bratelli diagram:

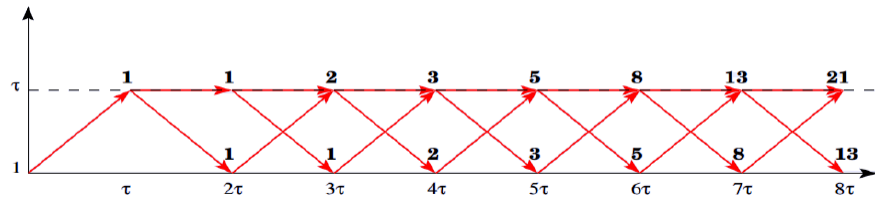


Figure 23: This is a Bratelli diagram. The number at the vertices show the number of ways one can get at that vertex if he starts from the origin.

which is to be translated as following: the number at the vertices represent the number of distinct connected paths one can follow to reach that vertex if he starts from the origin. Hence the name given to the model.

Other anyonic computational models have been developed over the passage of time but what makes Fibonacci anyonic model special is the fact that his model despite its simplicity exhibits the essential property for simulating a quantum computer, universality.

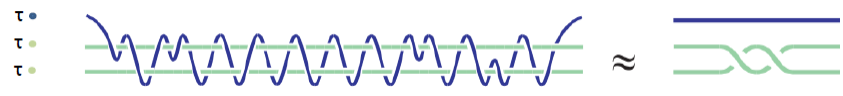
The first requirement in order to have universality is the fact that we have single qubit gates that can span $SU(2)$. This was proven by Solovay and Kitaev using a version of a brute force search algorithm they developed. Basically they managed to show that any arbitrary long braid i.e. any arbitrary unitary transformation in $SU(2)$, can be simulated by lots of the elementary ones and it can be simulated with an arbitrary accuracy ϵ .

The second requirement that one has in his disposal the CNOT gate in the context of this anyonic computational model was proven by Bonesteel and Hormozi a few years later. To prove this they had to make the following observations:

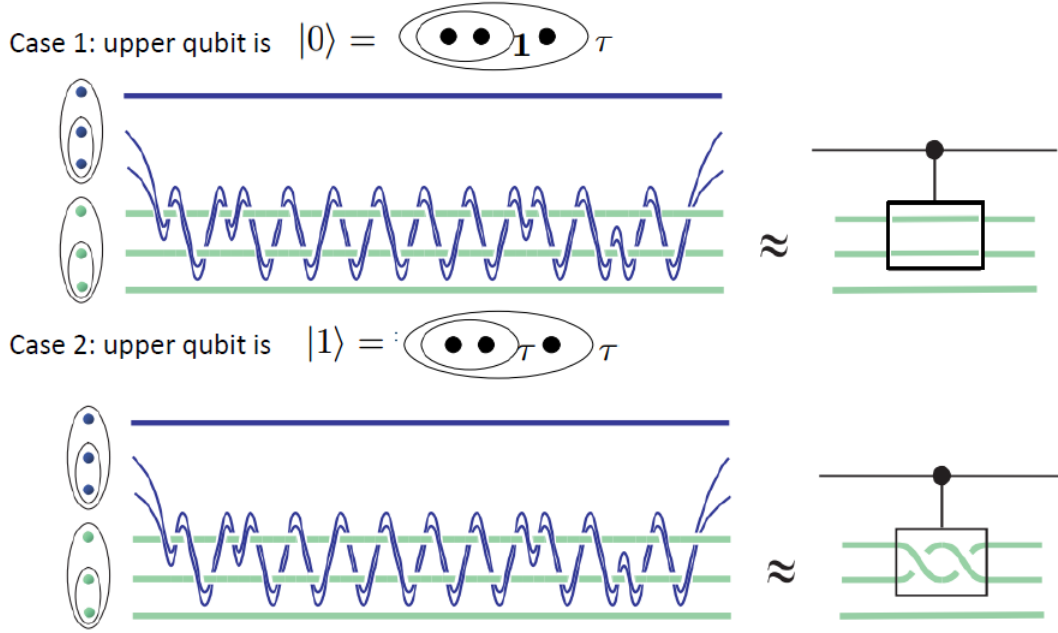
1. That if a Fibonacci anyon is winded around two other Fibonacci anyons in an appropriate manner, namely [3]:

$$\sigma_2^3 \sigma_1^2 \sigma_2^{-4} \sigma_1^2 \sigma_2^2 \sigma_1^{-2} \sigma_2^{-2} \sigma_1^{-2} \sigma_2^2 \sigma_1^2 \sigma_2^{-2} \sigma_1^2 \sigma_2^{-2} \sigma_1^4 \sigma_2^{-2} \sigma_1^2 \sigma_2^4 \sigma_1^2 \sigma_2^{-2} \sigma_2 \approx \sigma_1^2 \quad (29)$$

then this process is equivalent to leaving the first anyon unaffected and winding the other two around each other:



2. That if the first idea is implemented using the Fibonacci anyons that encode two logical qubits, then the CNOT gate can be simulated [3]. More specifically to do this consider winding a pair of anyons in a specific manner around two Fibonacci anyons rather than using just one anyon. Then depending on which logical state this pair belongs to then the effect of the process on the other qubit will be different as you can see on the figure below:



which is a 2-qubit gate equivalent to the CNOT gate we saw in the first chapter.

So in a summary of the above the Fibonacci anyonic model can be universal, and can achieve this with an arbitrary accuracy ϵ . Also it only requires $4n$ anyons to encode n logical qubits, so it is polynomial in resources required which is what one would like to have in order to characterize a model of computation efficient.

4.1 Fibonacci anyonic model and WZW models

There must exist a field theory describing these Fibonacci anyons. In order to find which is this theory, we first need to identify the group of symmetries that are related to the Fibonacci anyons, i.e. the group that encodes the fusion rules. This is not a trivial thing to do directly but what we have to do is to investigate the equivalency of the relations of the representations of the anyons. Essentially these group for Fibonacci anyons can be shown to be isomorphic to the even subgroup of the $SU(2)_3$ group [5]. This is understood by making the following observations:

1. Need to observe the similarity of the fusion rules of the Fibonacci anyons to the fusion rules for 1-spin particles if one cuts off spin-2 particle :

$$1 \otimes 1 = 0 \oplus 1$$

$$\tau \otimes \tau = 1 \oplus \tau$$

2. Then we need to note that it is possible to obtain a “quantized” version of the $SU(2)$ group, the $SU(2)_k$ by truncating the possible values of the angular momenta up to a maximum

angular momentum k with the in-between angular momenta taking values: $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, k$, e.g.

$$SU(2)_3 = \{0, \frac{1}{2}, 1, \frac{3}{2}\} \tag{30}$$

3. Then we have to consider the even part of $SU(2)_3$ which is basically a subgroup, isomorphic to the group of symmetries that encodes Fibonacci anyons fusion rules.

After discovering this group, we understand then that the field theory that is related to this group is the field theory describing the Fibonacci anyons. This theory is the $SU(2)_3$ WZW model coupled to a $U(1)$ gauge field [2]. In its low energy limit this theory reduces to a Chern-Simons theory which can be shown to describe particles that exhibit all the characteristics of anyons. Recently, Wang has constructed all anyons models with no more than four labels, and has found that all of the models are closely related to the models that are found in Chern-Simons theory.

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