Logarithmic Conformal Field Theory

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Contents

1 Basic facts about proper CFT

1.1 Preliminary considerations

It is well known, that symmetries are closely related to conserved quantities. The more symmetries exist within a given physical theory, the more conserved quantities can be derived to perform calculations within a certain framework. In the case of a quantum field theory, symmetries restrict the form of correlators or can even be used to completely determine the form of a correlator.
Often, when dealing with a quantum field theory, the appropriate symmetry group is given by the Poincaré group. However, in a conformal quantum field theory this group is enlarged and contains also scaling transformations and special conformal transformations. As was already mentioned in Henrik Dreyer's talk, one finds that the conformal group for $\mathbb{R}^{d,1}$ is isomorphic to $SO(d+1,1)$. Since this is a finite dimensional group, there are only finitely many conserved quantities and the corresponding CFT is not completely solvable (i.e. not all correlators are computable). The above result still holds for a field theory in $d = 2$ spatial dimensions with global conformal invariance and furthermore it can be proven that $SO(3,1) \simeq SL(2,\mathbb{C})$. The Möbius group is only three dimensional over $\mathbb{C}$ and thus the theory is not completely solvable. The analysis presented in Henrik Dreyer's talk, which led to the isomorphy between the conformal group and the special orthogonal group breaks down only for *local* conformal invariance in two spatial dimensions. A modified analysis leads to the conclusion that the appropriate symmetry group is provided by the group of all holomorphic functions. This group gives rise to an infinite number of conserved quantities and promotes two-dimensional CFT to an exactly solvable theory.
The next two sections are devoted to the tools needed to discuss global and local conformal invariance in two-dimensional conformal field theory. We will see, how these two types of conformal symmetry manifest themselves in the formalism of proper CFT.

1.2 Two-dimensional field theory with global conformal invariance

Imposing global conformal invariance on a two-dimensional field theory drastically restricts the allowed number of Virasoro modes. As was shown in Henrik Dreyer's talk, only the modes L_{-1}, L_0 and L_1 are compatible with the requirement of global conformal invariance. These three modes are the generators of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ and this means that they generate infinitesimal conformal transformations. The general transformation law for a primary field ϕ_h subjected to infinitesimal conformal transformations was derived in Daniel Herrs talk and reads

$$\phi_h \to \phi_h + \delta_n \phi_h = \phi_h + \left(z^{n+1}\partial_z + h(n+1)z^n\right)\phi_h \quad \text{with } n \in \mathbb{Z}. \tag{1}$$

The differential operator on the right hand side can be rewritten as the commutator of the primary field with the n-th Virasoro mode,

$$\phi_h \to \phi_h + [L_n, \phi_h] \quad \text{with } n \in \mathbb{Z}. \tag{2}$$

This notation emphasizes the fact, that infinitesimal conformal transformations are generated by the Virasoro modes and that these results are also true in the case of a theory with local conformal invariance. However, in the global case there are only three allowed Virasoro modes and n is restricted to the set $\{-1, 0, 1\}$.

To implement the requirement of global conformal invariance into our theory, we have to look at the N-point function $G_N := \langle \phi_{h_1}(z_1)\phi_{h_2}(z_2)\cdots\phi_{h_N}(z_N)\rangle$. Since every field appearing in the N-point function infinitesimally transforms as (1), the behavior of G_N is given by

$$G_N \to G_N + \delta_n G_N. \tag{3}$$

To construct a theory with global conformal invariance we have to demand $\delta_n G_N = 0$ for all points and for $n \in \{-1, 0, 1\}$. It is easy to see, using the differential operator appearing in (1), that this condition can be rewritten as a system of three partial differential equations of first order:

$$\begin{aligned}
L_{-1} \quad &: \qquad \sum_{i=1}^{N} \partial_{z_i} G_N(z_1, ..., z_N) = 0 \\
L_0 \quad &: \qquad \sum_{i=1}^{N} (z_i\partial_{z_i} + h_i) G_N(z_1, ..., z_N) = 0 \\
L_1 \quad &: \qquad \sum_{i=1}^{N} (z_i^2\partial_{z_i} + 2h_i z_i) G_N(z_1, ..., z_N) = 0.
\end{aligned} \tag{4}$$

These differential equations, also known as the global conformal Ward identities, restrict the form of the N-point function and are a manifestation of global conformal invariance:

The only compatible Virasoro modes are directly related to the only compatible infinitesimal conformal transformations which finally give rise to the restricting system of partial differential equations. Solving the system for the two- and three-point function provides the same solution as we have seen in Daniel Heer's talk. As a reminder, we found that the two-point function is completely determined by global conformal invariance (up to an irrelevant field normalization factor) and the form of the three-point function is determined up to a constant factor. The form of higher order functions is less restricted by global invariance and they are only known up to an arbitrary (but conformal invariant) function. For example, solving the conformal Ward identities for the four point functions leads to

$$\langle \phi_{h_1}(z_1)\phi_{h_2}(z_2)\phi_{h_3}(z_3)\phi_{h_4}(z_4)\rangle = F(x) \prod_{i>j} (z_i - z_j)^{\frac{\sum_{k=1}^{4} h_k}{3} - h_i - h_j} \tag{5}$$

where $x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$ is the so called anharmonic ratio and $F(x)$ is the above mentioned arbitrary function[2].

1.3 Two-dimensional field theory with local conformal invariance

Two-dimensional field theories which posses a local conformal symmetry differ very much from their global counterparts. In the case of local conformal symmetry, the appropriate symmetry group is provided by the group of all holomorphic functions and is thus infinite dimensional. This implies that all Virasoro modes L_n with $n \in \mathbb{Z}$ lead to allowed infinitesimal conformal transformations of the form (1) and there are infinitely many conserved quantities. Put differently: Based on our discussion of the global case we may expect that infinitely many Virasoro modes lead to constraining equations for the N-point function.

However, the requirement of local conformal invariance is not so easy to implement. In the global case we demanded $\delta_n G_N(z_1, ..., z_N) = 0$ for all points $z_i \in \mathbb{C}$, but this is not possible in the local case since we can't possibly expect that $\delta_n G_N(z_1, ..., z_N) = 0$ is true on whole $\mathbb{C}$. Moreover, how would we define "local" and how would we adopt the global condition for the local case? What happens if all but one points are in a "local neighborhood"?

The difficulty of finding conserved quantities in the case of a field theory with local conformal symmetry can be overcome by having a closer look at the algebraic properties of the theory. In the second talk of the Proseminar, when we studied the representation theory of proper CFTs, we learned that there is a highest weight state (HWS) $|h,c\rangle$ and the action of the Virasoro modes on the HWS creates new states. More precisely, we saw that $L_{|n|}|h,c\rangle = 0$ and only $L_{-|n|}|h,c\rangle$ leads to new states. These so called descendant states can be organized in Verma modules[2]:

$$\mathcal{V}_{h,c} := \text{span}\left\{ \prod_{n_i \in I} L_{-n_i}|h,c\rangle \middle| I = \{n_1, ..., n_k\} \subset \mathbb{N} \ , \ n_{i+1} \geq n_i \right\}. \qquad (6)$$

The first condition implies the negativity of the indices $-n_i$ and the second condition implements the natural graduation of the Verma module due to the Virasoro algebra. It is convenient to rewrite the Verma module as[2]

$$\mathcal{V}_{h,c} = \bigoplus_N \mathcal{V}_{h,c}^{(N)} \qquad (7)$$

where $N = \sum_{i=1}^k n_i$ is the level of the Verma module and $\mathcal{V}_{h,c}^{(N)}$ is given by

$$\mathcal{V}_{h,c}^{(N)} = \text{span}\left\{ \prod_{n_i \in I} L_{-n_i}|h,c\rangle \middle| I = \{n_1, ..., n_k\} \subset \mathbb{N} \ , \ \sum_{i=1}^k n_i = N \right\}. \qquad (8)$$

A closer look to the level N Verma module leads to a natural question: Are all states $|\lambda_{h,c}^{(N)}\rangle \in \mathcal{V}_{h,c}^{(N)}$ for a given level N linearly independent?
We have seen in Stephanie Mayer's talk, that this question can be rephrased as: Is there a state $|\lambda_{h,c}^{(N)}\rangle \in \mathcal{V}_{h,c}^{(N)}$ with $\langle\psi|\lambda_{h,c}^{(N)}\rangle \equiv 0 \ \forall|\psi\rangle \in \mathcal{H}$?
In Stephanie Mayer's talk we learned that such states exist and she developed the general formalism to find these states at any level. We wont go through the whole formalism again, but we will discuss the very important example of level two degeneracy in some detail. The discussion will basically follow the structure presented in subsection 2.3 of reference [2].
To begin with, we note that the Verma module $\mathcal{V}_{h,c}^{(N)}$ for the $N = 2$ case contains only linear combinations of the operators L_{-2} and L_{-1}^2. The most general null vector in this Verma module can be denoted as $|\lambda_{h,c}^{(2)}\rangle = \left(L_{-2} + aL_{-1}^2\right)|h,c\rangle$. If this vector is really a null vector, it satisfies two conditions (where the second condition one is actually a consequence of the first condition):

1. $\langle\psi|\lambda_{h,c}^{(2)}\rangle \equiv 0 \ \forall|\psi\rangle \in \mathcal{H}$.

2. The determinant of the 2×2 Gram matrix $K^{(2)}$ is zero.

The Gram matrix is simply given by

$$K^{(2)} = \begin{pmatrix} \langle h,c|L_2L_{-2}|h,c\rangle & \langle h,c|L_2L_{-1}^2|h,c\rangle \\ \langle h,c|L_1^2L_{-2}|h,c\rangle & \langle h,c|L_1^2L_{-1}^2|h,c\rangle \end{pmatrix} \tag{9}$$

and the matrix elements can easily be computed using the Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}$. For the first matrix element we get for example

$$\begin{aligned} L_2L_{-2}|h,c\rangle &= L_{-2}L_2|h,c\rangle + 4L_0|h,c\rangle + \frac{c}{12}(2^3-2)|h,c\rangle \\ &= 4h|h,c\rangle + \frac{c}{2}|h,c\rangle. \end{aligned} \tag{10}$$

Performing analogue computations for the remaining three elements we end up with

$$K^{(2)} = \begin{pmatrix} 4h+\frac{c}{2} & 6h \\ 6h & 4h+8h^2 \end{pmatrix} \langle h,c|h,c\rangle \tag{11}$$

and the determinant is found to be

$$\det(K^{(2)}) = 2h\left(16h^2 + 2(c-5)h + c\right)\langle h,c|h,c\rangle^2. \tag{12}$$

According to the first condition, this determinant has to vanish. Obviously, there are three solutions to the resulting cubic equation and we immediately see that the first one is given by $h=0$. However, $h=0$ is the solution one finds for the null vector at first level and because $L_{-1}|h,c\rangle = 0$ there is no null vector at second level of the form $\left(L_{-2} + aL_{-1}^2\right)|h,c\rangle$. The other two solutions can be found by solving the quadratic equation

$$16h^2 + 2(c-5)h + c = 0 \;\Rightarrow\; h = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right). \tag{13}$$

We conclude, that for a fixed central charge c there are only two admissible values for the conformal weight h such that the corresponding Verma module $\mathcal{V}_{h,c}^{(2)}$ contains a level two null vector.

In order to determine the factor a appearing in $|\lambda_{h,c}^{(2)}\rangle = \left(L_{-2} + aL_{-1}^2\right)|h,c\rangle$ we have to satisfy the second condition. The Hilbert space $\mathcal{H}$ can be constructed as direct sum of Verma modules: $\mathcal{H} = \bigoplus_h \mathcal{V}_{h,c}$. Clearly, this space contains an infinite amount of vectors, which may also assume a rather complicated form. However, it is also clear that all but two types of vectors are orthogonal to $|\lambda_{h,c}^{(2)}\rangle$ and we only need to consider vectors of the form $L_{-1}|\alpha\rangle$

and $L_{-2}|\beta\rangle$. The first inner product is quickly computed using again the Virasoro algebra.

$$\begin{aligned}\langle\alpha|L_1|\lambda_{h,c}^{(2)}\rangle &= \langle\alpha|L_1\left(L_{-2}+aL_{-1}^2\right)|h,c\rangle \\ &= (3+2a(2h+1))\,\langle\alpha|L_{-1}|h,c\rangle \overset{!}{=} 0 \\ &\Rightarrow\ a = -\frac{3}{2(2h+1)}.\end{aligned} \tag{14}$$

Finally, the second inner product (together with the expression for a) leads to an equation which relates the conformal weight h to the central charge c.

$$\begin{aligned}\langle\beta|L_2|\lambda_{h,c}^{(2)}\rangle &= \langle\beta|L_2\left(L_{-2}+aL_{-1}^2\right)|h,c\rangle \\ &= \left(4h+\frac{c}{2}+6ah\right)\langle\beta|h,c\rangle \overset{!}{=} 0 \\ &\Rightarrow c = 2h\frac{5-8h}{2h+1}.\end{aligned} \tag{15}$$

Considering both results obtained in exploring the two conditions for null vectors we learn that there can be a level two null vector only if the conformal weight h assumes one of the two calculated values and satisfies equation (15). When both requirements are met, the null vector takes the form

$$|\lambda_{h,c}^{(2)}\rangle = \left(L_{-2}-\frac{3}{2(2h+1)}L_{-1}^2\right)|h,c\rangle. \tag{16}$$

The crucial point now is that a null vector completely decouples from all other states in the Hilbert space. Hence, a null vector inserted into a correlator inevitably yields zero. And since a null vector is only build up from Virasoro modes and these modes can be translated into differential operators, we finally achieve our goal. The fact that we have an identity holding (null vector in correlator equals zero) and the translation of the Virasoro modes to differential operators allows us to write down differential equations for correlators. That's how local conformal invariance manifests itself. And it also matches our intuition. As was already mentioned, we expect that in the case of local conformal invariance infinitely many Virasoro modes lead to constraining equations for correlators. Since there are infinitely many null vectors (at higher and higher degeneracy level) there are also infinitely many differential equations, which can be derived. We recognize that a CFT with local conformal invariance is in principle exactly solvable and that this is due to the fact that the symmetry group is infinite dimensional.
To conclude this section and for the sake of completeness we quickly derive

the differential equation belonging to the level two null vector. For this purpose we use the well-known relation

$$\langle \Phi^{(-r_1,\ldots,-r_k)}(z)X\rangle = \mathcal{L}_{-r_1}(z)\cdots\mathcal{L}_{-r_k}(z)\langle\Phi(z_0)X\rangle \tag{17}$$

where $X = \Phi_{h_1}(z_1)\cdots\Phi_{h_N}(z_N)$ denotes a product of $N-1$ primary fields Ψ_{h_i} with conformal weights h_i and the translation of the n-th Virasoro mode to a differential operator reads

$$\mathcal{L}_{-n}(z) = \sum_{i=1}^{N-1}\left(\frac{(n-1)h_i}{(z_i-z)^n} - \frac{1}{(z_1-z)^{n-1}}\frac{\partial}{\partial z_i}\right). \tag{18}$$

In our case this relations yield

$$\left[\sum_{i=1}^{N-1}\left(\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right) - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right]\langle\Phi(z)X\rangle = 0 \tag{19}$$

as constraining differential equation for an arbitrary N-point function.

2 Following the historical path

The last chapter was devoted to two aims. On one side it offers a summary of important properties of proper two-dimensional CFT, which were mostly developed in preceding talks and it allows us to directly compare proper CFT with logarithmic CFT. On the other side, it provides a good preparation to understand Gurarie's original approach to LCFT presented in [1] which will be our guideline.

2.1 Setting up the theory framework

Historically, Gurarie was interested in studying a CFT with central charge $c = -2$. Such CFTs often appear in applications and already before Gurarie's work it was understood, that in some $c = -2$ CFTs the conventional formalism breaks down.
Since the form of the two- und three-point functions is determined by the conformal Ward identities, Gurarie tackled the task to compute the four-point function from local conformal invariance. This can be achieved by the use of null vectors.
At first level we find the null vector $L_{-1}|h,c\rangle$, recover the third Ward identity and we don't gain any additional information. Fortunately, we found a

level two null vector $\left(L_{-2} + aL_{-1}^2\right)|h,c\rangle$ and a non-trivial differential equation corresponding to it in the previous section.
These simple requirements define the framework of the theory we want to study:

1. The central charge is set to be $c = -2$.

2. According to equation (13), the requirement of a level two null vector implies two possible values for the conformal weight: $h \in \{-\frac{1}{8}, 1\}$. Following Gurarie we choose $h = -\frac{1}{8}$. Thus, our theory contains a primary field μ of conformal weight $h_\mu = -\frac{1}{8}$ which is degenerate to the second level.

3. From (16) we find $|\lambda_{h_\mu,c}^{(2)}\rangle = \left(L_{-2} - 2L_{-1}^2\right)|h_\mu, c\rangle$ as null vector. Note by the way that the above definition of the primary field μ implies $\mu|0\rangle = |h_\mu, c\rangle$ and thus $|\lambda_{h_\mu,c}^{(2)}\rangle = \left(L_{-2} - 2L_{-1}^2\right)\mu|0\rangle$. This property is crucial for the next subsection.

2.2 Computing the four-point function

Having defined the framework of our theory we can finally move on to the actual task: Compute the four-point function of four equal primary fields. Using (20) we can write the correlator as

$$\langle \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4)\rangle = (z_1 - z_3)^{\frac{1}{4}}(z_2 - z_4)^{\frac{1}{4}}\left[x(1-x)\right]^{\frac{1}{4}} F(x) \tag{20}$$

where we also redefined $F(x)$ to be $\left[x(1-x)\right]^{\frac{1}{4}} F(x)$. Since the prefactor only depends on the anharmonic ratio, it is conformal invariant and there's nothing bad about extracting this factor from $F(x)$. As we will see later, it is convenient to do so because this prefactor simplifies some calculations.
The next thing to do is of course to insert the null vector into the correlator and to derive the corresponding differential equation:

$$\begin{aligned}
&\left\langle \mu(z_1)\mu(z_2)\mu(z_3)\left(L_{-2} - 2L_{-1}^2\right)\mu(z)\right\rangle = 0 \\
\Rightarrow &\left[\sum_{i=1}^{3}\left(\frac{1}{z-z_i}\frac{\partial}{\partial z_i} - \frac{1}{8}\frac{1}{(z-z_i)^2}\right) - 2\frac{\partial^2}{\partial z^2}\right]\langle \mu(z_1)\mu(z_2)\mu(z_3)\mu(z)\rangle = 0.
\end{aligned} \tag{21}$$

The obtained differential equation looks pretty complicated since it involves derivatives and rational prefactors in all four points z_i.

Remember however, that we extracted a useful prefactor of the form $[x(1-x)]$ from the function $F(x)$. It's this prefactor which allows to recast the differential equation into the form

$$x(1-x)\frac{d^2F(x)}{dx^2} + (1-2x)\frac{dF(x)}{dx} - \frac{1}{4}F(x) = 0 \tag{22}$$

where only derivatives and polynomial prefactors in x appear. The new differential equation is much easier and it can be solved using Frobenius method. According to Frobenius, the most general ansatz to solve equation (22) is provided by

$$F(x) = x^s \sum_{n=0}^{\infty} a_n x^n \quad \text{with } a_0 \neq 0. \tag{23}$$

This ansatz introduces a new parameter, s, which has to be determined before we can procede in calculating the coefficents a_n. Since a_0 is defined to be the first non-zero coefficent of the power series, we can use the following procedure to determine s: Plug the ansatz into the differential equation (22), group the coefficents by power and start comparing them. This leads to an expression of the form $f(s,n)a_0 = 0$ which implies $f(s,n) = 0$. By solving this equation (the so-called indicial equation), we find the possible values of the new variable s.
In our case we get

$$[s(s-1)+s]\, a_0 = 0 \Rightarrow s^2 = 0 \tag{24}$$

from which we conclude that both roots of the indicial equation are zero. From a qualitative point of view we would deduce from the fact that there are two solutions to the indicial equation, say $s_1 = 2$ and $s_2 = \sqrt{2}$, that there are two linearly independent solutions to the differential equation. Conversely, we would expect that in our case there is only one solution to equation (22). Unfortunately, this time our intuition is wrong. Indeed, despite the fact that $s_1 = s_2 = 0$, there are two linearly independent solutions. Actually, there is a deeper mathematical reason for having two linearly independent solutions and the discussed differential equation is only a special case belonging to a whole class of differential equations which are governed by a theorem known as Fuch's theorem. However, we want to continue our discussion using only basic analytical tools which don't ask for special knowledge in the theory of differential equations.
Before we can gain any deeper insight, we have to find the recursion relation for the coefficients a_n. Now that we know the value of s we immediately

find

$$
\begin{aligned}
&\sum_{n=0}^{\infty} a_n n^2 x^{n-1} - \sum_{n=0}^{\infty} \left(a_n n(n-1) + 2a_n n + \frac{1}{4} a_n \right) \\
&= \sum_{n=0}^{\infty} a_n n^2 x^{n-1} - \sum_{n=0}^{\infty} a_n \left(n^2 + n + \frac{1}{4} \right) x^n \\
&= \sum_{n=0}^{\infty} a_n n^2 x^{n-1} - \sum_{n=0}^{\infty} a_n \left(n + \frac{1}{2} \right)^2 x^n = 0
\end{aligned} \tag{25}
$$

and simply by adjusting the second sum with $n \to n-1$ and comparing coefficents, we get the recursion relation

$$
a_n n^2 - a_{n-1} \left(n - \frac{1}{2} \right)^2 = 0. \tag{26}
$$

By writing down the first few coefficents one recognizes that the recursion relation is solved by

$$
a_n = a_0 \left[\frac{(2n-1)!!}{(2n)!!} \right]^2 \tag{27}
$$

which finally leads to the first solution of the differential equation:

$$
F(x) = a_0 \sum_{n=0}^{\infty} \left[\frac{(2n-1)!!}{(2n)!!} \right]^2 x^n. \tag{28}
$$

Using the well-known formula $r = \lim_{n\to\infty} |a_n/a_{n+1}|$ for the radius of convergence, we find that the power series is finite only for $x \in D = \big\{ x \in \mathbb{C} \big| |x| < 1 \big\}$. Fortunately, the power series in (28) is a well-known expansion. It can be identified (up to a constant factor) as the expansion of the analytic continuation of the elliptic integral of the first kind in the unit disk. This is certainly not an obvious fact but it can easily be checked and once we accepted it, we can substitute the power series by an integral:

$$
F(x) \propto G(x) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1 - x \sin^2(\theta)}}. \tag{29}
$$

As we noted before, the power series (28) behaves well in the unit disc and diverges for $|x| \geq 1$. A look at the elliptic integral reveals that it converges also for values of x which lie outside of the unit disc. Indeed, the integral converges on whole $\mathbb{C}$, exept for points where $x \sin^2(\theta) - 1 = 0$. Obviously, this can only happen for $|x| = 1$ and $x \in \mathbb{R}$. From the two possible solutions to this problem we can immediately discard $x = -1$ since $\sin^2(\theta) \geq 0$. The

solution $x = 1$ leads to the equation $\sin^2(\theta) - 1 = 0$ which is solved by $\theta = \frac{\pi}{2}$. Due to the fact that this solution lies in the integration interval $\left[0, \frac{\pi}{2}\right]$, we get a divergency in $x = 1$.
To find out with what type of divergence we have to deal with, we set $x = 1$ and compute the integral

$$\begin{aligned} G(1) &= \lim_{\alpha \to \frac{\pi}{2}} \int_0^\alpha \frac{\mathrm{d}\theta}{\sqrt{1 - \sin^2(\theta)}} = \lim_{\alpha \to \frac{\pi}{2}} \int_0^\alpha \frac{\mathrm{d}\theta}{\cos(\theta)} \\ &= \lim_{\alpha \to \frac{\pi}{2}} \ln \left| \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \right| \Bigg|_0^\alpha \to \infty. \end{aligned} \tag{30}$$

Hence, we learn that $\lim_{x \to 1} G(x)$ is logarithmically divergent and we complete our discussion of the first solution by noting that now we basically know how it behaves and therefore we also know the behavior of the four-point function.
However, this is not the end of the story. A closer look at the differential equation (22) reveals that there is a second solution. Indeed, for symmetry reasons also $G(1 - x)$ has to be a solution. Since $G(x)$ diverges logarithmically for $|x| \to 1$, $G(1 - x)$ has to diverge in the same way for $|x| \to 0$. Interestingly, this fact allows us to learn more about $G(1 - x)$: Choose as ansatz

$$G(1 - x) = \log(x) \sum_{n=0}^{\infty} b_n x^n + \sum_{n=0}^{\infty} c_n x^n. \tag{31}$$

The logarithm explicetly produces the wanted behavior for $|x| \to 0$ and the two power series are finite in $x = 0$ by construction. By plugging in this ansatz into the differential equation, we find that the coefficents b_n are the same as the coefficents a_n from the first solution. Hence,

$$G(1 - x) = \log(x) G(x) + H(x) \tag{32}$$

where $H(x)$ is some analytic (but for our discussion unimportant) function. The important thing is that the second solution can be expressed using the first solution and that the two solutions are manifestly linear independent from each other. Of course, the full solution to (22) is given by the linear combination

$$F(x) = AG(x) + BG(1 - x) \tag{33}$$

which makes clear, that we can't get rid of the logarithmic divergence. This is a very important statement and the consecutive subsections will be dedicated to the consequences related to the logarithmic divergencies.

2.3 Reinterpretation of the four-point function as OPE

In the previous section we found as solution to the differential equation a power series, which converges only in the unit disc. This is due to the presence of a singularity on the boundary of the unit disc. Fortunately, we regocnized the power series as expansion of the analytic continuation of the elliptic integral of the first kind. The integral representation allowed us to identify the singularity in $x = 1$ and to extend the domain of the solution from D to $\mathbb{C}\backslash\{1\}$. Moreover, we learned that the full solution (33) of the differential equation always displays a logarithmic divergence. This gives rise to the natural question: Is there something bad about this type of divergence?

Remember that also other correlators, as for example the two- and three-point function, show divergent behavior. Divergencies are therefore nothing new. But one may object that logarithms aren't conformal invarant functions and we therefore constructed a correlator which violates conformal invariance and something must have gone terribly wrong. We invalidate this objection by observing that the logarithm appearing in $G(1-x)$ depends on the anharmonic ratio. Since the argument is conformal invariant, the logarithm itself is conformal invariant, too.

The first problems caused by the logarithmic behavior appear when we try to reinterprete the four-point function as operator product expansion. According to [3], the most general expression for an OPE of two primary fields $\phi_{h_n}(z)\phi_{h_m}(0)$ can be written as

$$\begin{aligned} \phi_{h_n}(z)\phi_{h_m}(0)|0\rangle &= \sum_p \sum_{\{k\}} C_{nm}^{p,\{k\}} z^{h_p - h_m - h_n + \sum_i k_i} \phi_p^{\{k\}}(0)|0\rangle \\ &= \sum_p C_{nm}^{p} z^{h_p - h_m - h_n + \sum_i k_i} \Psi_p(z|0)|0\rangle \end{aligned} \tag{34}$$

where $\phi_p^{\{k\}}$ denotes descendant fields belonging to the conformal family $[\phi_p]$ and $\Psi_p(z|0) = \sum_{\{k\}} \beta_{nm}^{p,\{k\}} z^{\sum_i k_i} \phi_p^{\{k\}}(0)$ is introduced to split up summation over conformal families (sum over p) and summation over the level of the descendant fields (sum over $\{k\}$). To achieve the splitting up we also have to introduce the $\{k\}$-independent constants C_{nm}^p defined by $C_{nm}^{p,\{k\}} = C_{nm}^p \beta_{nm}^{p,\{k\}}$. In our specific case, expression (34) simplifies a lot since we are interested in the OPE of $\mu(z)\mu(0)|0\rangle$. This implies $h_m = h_n = -\frac{1}{8}$. Moreover, we can restrict ourselves to the $h_p = 0$ case. This is not an approximation, it is shown in [3] that the indicial equation $\alpha^2 = 0$ implies the existence of two operators with conformal weight $h_p = 0$ (two identity operators which coincide in each point z). This means, there are two OPEs which involve only the conformal family of the identity operator $\mathbf{I}$ (hence, both OPEs are equal). Consequently, the sum over p drops out and $h_p = 0$. Finally, we end up with ($C^p_{nm} \equiv C$)

$$\begin{aligned}\mu(z)\mu(0)|0\rangle &\sim Cz^{-2h_\mu}\Psi(z|0)|0\rangle = z^{-2h_\mu}\sum_{\{k\}} z^{\sum_i k_i} C\beta^{\{k\}}\mathbf{I}^{\{k\}}(0)|0\rangle \\ &= z^{\frac{1}{4}}\sum_{k=0}^{\infty} z^k \underbrace{C\beta^{\{k\}}\mathbf{I}^{\{k\}}(0)|0\rangle}_{=:|\mathbf{I},k\rangle} = z^{\frac{1}{4}}\underbrace{\sum_{k=0}^{\infty} z^k |\mathbf{I},k\rangle}_{=:|\mathbf{I},z\rangle}\,.\end{aligned} \tag{35}$$

At first sight, the result $\mu(z)\mu(0)|0\rangle \sim z^{\frac{1}{4}}|\mathbf{I},z\rangle$ looks as if it can reproduce the four-point function. By applying (35) on the four-point function we produce a factor $z^{\frac{1}{4}}w^{\frac{1}{4}}$, just as in (20). The product $\langle \mathbf{I},l|\mathbf{I},k\rangle$ should then account for the rest of the four-point function and in particular, it should reproduce the logarithmic divergence. However, it is impossible to create the z-dependent logarithm $\log(z)$ appearing in the second solution of the differential equation just by using the OPE (35). Moreover, it can be proved, that our theory together with the usual OPE (34) violates the crossing symmetry[2]. The validity of the crossing symmetry condition (see [3] for a detailed discussion on this condition) can also be seen as a self-consistency condition of every conformal field theory since it simply reflects the associativity of the operator product.
The fact that the usual OPE applied on our theory fails to explain the logarithmic behavior of the four-point function and that it violates the crossing symmetry condition tells us two things:

1. Our theory is not self-consistent.
2. If there is any chance to make our theory self-consistent we have to modify the usual OPE. The OPE is the source of all problems.

This two facts were first recognized by Gurarie and he proposed the following modification of the OPE:

$$\begin{aligned}&\mu(z)\mu(0)|0\rangle = z^{\frac{1}{4}}|\mathbf{I},z\rangle \quad \text{(consistent with 1}^{\text{st}}\text{ solution)} \\ &\mu(z)\mu(0)|0\rangle = z^{\frac{1}{4}}\left(\log(z)|\mathbf{I},z\rangle + |\mathbf{I}_1,z\rangle\right) \text{(consistent with 2}^{\text{nd}}\text{ solution)}\end{aligned} \tag{36}$$

(Actually, the second solution is the only correct one and everything that follows could be derived using the second solution only. However, we will

stick to Gurarie's notation since it simplifies the discussion.)
As was already mentioned, the fusion rules developed in [3] imply that the OPE involves two operators of conformal weight $h_p = 0$ (identity operators) which coincide. So, in the usual OPE the expression corresponding to the first solution and the one corresponding to the second solution are equal and they fail to explain the properties of the four-point function. In Gurarie's modification, the forms of the OPEs already resemble the forms of the two solutions and there are still two operators of conformal weight $h_p = 0$, but they no longer coincide. This modification promotes our ill-defined theory to a self-consistent theory, but the price we have to pay is the introduction of a new kind of operator. As we will discover, the so-called logarithmic partner $\mathbf{I}_1$ of the primary field $\mathbf{I}$ has some unusual properties which will modify many aspects of our theory.

2.4 The unusual properties of the logarithmic partner

Originally, we started with a special $c = -2$ CFT which contained only one (non-trivial) operator. However, we have seen in the last subsection, that the primary field μ is not enough to construct a self-consistent theory based on the OPE. Consistency required from us to introduce a new kind of operator. Though, a priori it's not clear that $\mathbf{I}_1$ isn't a primary field. In order to see it and to explore the properties of $\mathbf{I}_1$ we will consider the action of the Virasoro modes $L_{n\geq 0}$ on both OPEs (36) and calculate the results in two different ways. We have seen in (35) that

$$|\mathbf{I}, z\rangle = \sum_{k=0}^{\infty} z^k |\mathbf{I}, k\rangle \tag{37}$$

and we already argued, that due to the fusion rule and Gurarie's modification there should be two identity operators, which aren't equal. This justifies the ansatz

$$|\mathbf{I}_1, z\rangle = \sum_{k=0}^{\infty} z^k |\mathbf{I}_1, k\rangle. \tag{38}$$

These two expansions together with the commutation relation of L_n with the primary field μ (also a result of proper CFT which we often encountered in previous talks)

$$[L_n, \mu(z)] = \left(z^{n+1} \frac{\partial}{\partial z} + (n+1) h_\mu z^n \right) \tag{39}$$

are needed to evaluate $L_n\mu(z)\mu(0)|0\rangle$ for $n \geq 0$. Let's start with the $n = 0$ case and the first OPE in (36):

$$\begin{aligned} L_0\mu(z)\mu(0)|0\rangle &= \left(z\frac{\partial}{\partial z}\left(\mu(z)\mu(0)\right) + h_\mu\mu(z)\mu(0) \right)|0\rangle + \mu(z)\underbrace{L_0\mu(0)|0\rangle}_{=h_\mu\mu(0)|0\rangle} \\ &= \sum_{k=0}^{\infty}(k-2h_\mu)\,z^{k-2h_\mu}|\mathbf{I},k\rangle + 2h_\mu\mu(z)\mu(0)|0\rangle \\ &= \sum_{k=0}^{\infty} kz^{k-2h_\mu}|\mathbf{I},k\rangle \overset{!}{=} \sum_{k=0}^{\infty} z^{k-2h_\mu}L_0|\mathbf{I},k\rangle \\ &\Rightarrow L_0|\mathbf{I},k\rangle = k|\mathbf{I},k\rangle. \end{aligned} \tag{40}$$

On the first line we simply use the commutation relation and the fact that $\mu(0)|0\rangle$ is the HWS of the theory. To actually evaluate the derivative of the product we insert the first OPE of (36), use expansion (37) and simplify the result on the third line (the terms proportional to h_μ cancel). The final result has to be equal to L_0 directly applied on the OPE. This leads to the eigenvalue equation on the last line of (40) which can be obtained in any conventional CFT.

The evaluation of $L_0\mu(z)\mu(0)|0\rangle$ for the second OPE is performed in nearly the same way:

$$\begin{aligned} &L_0\mu(z)\mu(0)|0\rangle \overset{1)}{=} \\ &\quad z\frac{\partial}{\partial z}\left(\log(z)\sum_{k=0}^{\infty} z^{k-2h_\mu}|\mathbf{I},k\rangle + \sum_{k=0}^{\infty} z^{k-2h_\mu}|\mathbf{I}_1,k\rangle \right) + kh_\mu\mu(z)\mu(0)|0\rangle \\ &\overset{2)}{=} \sum_{k=0}^{\infty} z^{k-2h_\mu}|\mathbf{I},k\rangle + \log(z)\sum_{k=0}^{\infty}(k-2h_\mu)z^{k-2h_\mu}|\mathbf{I},k\rangle \\ &\quad + \sum_{k=0}^{\infty}(k-2h_\mu)z^{n-2h_\mu}|\mathbf{I}_1,k\rangle + \sum_{k=0}^{\infty} z^{k-2h_\mu}\left(2h_\mu\log(z)|\mathbf{I},k\rangle + 2h_\mu|\mathbf{I}_1,k\rangle\right) \\ &\overset{3)}{=} \sum_{k=0}^{\infty} z^{k-2h_\mu}|\mathbf{I},k\rangle + \log(z)\sum_{k=0}^{\infty} z^{k-2h_\mu}k|\mathbf{I},k\rangle + \sum_{k=0}^{\infty} kz^{k-2h_\mu}|\mathbf{I}_1,k\rangle \\ &\overset{!}{=} \log(z)\sum_{k=0}^{\infty} z^{k-2h_\mu}\underbrace{L_0|\mathbf{I},k\rangle}_{=k|\mathbf{I},k\rangle} + \sum_{k=0}^{\infty} z^{k-2h_\mu}L_0|\mathbf{I}_1,k\rangle \\ &\Rightarrow L_0|\mathbf{I}_1,k\rangle = |\mathbf{I},k\rangle + k|\mathbf{I}_1,k\rangle. \end{aligned} \tag{41}$$

In 1) we use again the commutation relation (39) and the second OPE of (36) to evaluate the derivative of the product $\mu(z)\mu(0)|0\rangle$. The result of this computation is shown in 2) and we recognize that some terms cancel (indicated be the red and blue slashes). Finally, the simplified expression in 3) has to be equal to the direct application of L_0 on the OPE. After using the relation $L_0|\mathbf{I}, k\rangle = k|\mathbf{I}, k\rangle$, which we found in (40), we see that there are two cancelling terms (green slashes). By comparing coefficients we find the first unusual property of the logarithmic partner.
There are additional properties for the primary field and its logarithmic partner which follow from analyzing $L_{n>0}\mu(z)\mu(0)|0\rangle$. Since the calculations for these cases are rather lengthy, but are performed in an analogous way, we won't explicitly derive the properties

$$L_n|\mathbf{I}, n+k\rangle = (k + (n-1)h_\mu)\,|\mathbf{I}, k\rangle \tag{42}$$

and

$$L_n|\mathbf{I}_1, n+k\rangle = |\mathbf{I}, k\rangle + (k + (n-1)h_\mu)\,|\mathbf{I}_1, k\rangle. \tag{43}$$

Equation (42) is again valid for any conventional CFT and equation (43) holds only for theories containing a logarithmic partner. As was already mentioned, the unusual properties of the logarithmic partner will modify many aspects of conventional CFT. In the next chapter we will discuss the modifications and compare them with the usual CFT formalism.

3 Comparison of proper CFT and logarithmic CFT

The formalism of conventional CFT was developed step by step in preceding talks and we learned about many aspects such as HWS, null vectors and conformal Ward identities. To properly compare LCFT with CFT, we should develop the LCFT formalism up to the point where we developed the CFT formalism. Since this goes beyond the scope of this rewiev, we will skip the details of the derivations and simply state the differences of the two theories, focus on the consequences of the unusual LCFT properties and refer to the relevant literature.

3.1 Representation theory

The starting points of our comparison are the properties of conventional CFT we derived in the previous chapter

$$\begin{aligned} L_0|\mathbf{I}, k\rangle &= k|\mathbf{I}, k\rangle \\ L_n|\mathbf{I}, n+k\rangle &= (k + (n-1)h_\mu)\,|\mathbf{I}, k\rangle \end{aligned} \tag{44}$$

and its unusual counterparts

$$\begin{aligned} L_0|\mathbf{I}_1,k\rangle &= |\mathbf{I},k\rangle + k|\mathbf{I}_1,k\rangle \\ L_n|\mathbf{I}_1,n+k\rangle &= |\mathbf{I},k\rangle + (k+(n-1)h_\mu)\,|\mathbf{I}_1,k\rangle \end{aligned} \tag{45}$$

which only appear in logarithmic CFTs. By comparing these two sets of identities we see that the logarithmic identities contain an additional $|\mathbf{I},k\rangle$ term and thus combine contributions from the primary operator and the logarithmic partner.

This superficial observation has actually important implications when we turn to representation theory. We immediately see that $L_0|\mathbf{I},k\rangle = k|\mathbf{I},k\rangle$ implies that L_0 is diagonalizable. However, the logarithmic counterpart contains an additional term, which produces off-diagonal terms. Hence, L_0 can't be diagonalized in the LCFT framework and the simplest possible form of L_0 is of Jordan block form:

$$L_0 \sim \begin{pmatrix} M_1 & M_2 \\ 0 & M_3 \end{pmatrix}. \tag{46}$$

The M_i represent non-trivial submatrices and their form and the way how they are computed can be found in any text book about linear algebra. What matters for our purpose isn't the exact content of this matrices, but the general structure of the matrix (46). It is known from representation theory, that matrices of the form (46) lead to reducible but indecomposable representations. The situation so far was a different one: We found the representations of the Virasoro algebra to be reducible and decomposable. This means, that we can find irreducible subrepresentations ρ_i of the representation ρ of the Virasoro algebra (reducibility) and that we can write $\rho = \bigoplus_i \rho_i$ (decomposability). In other words: If we know the building blocks of ρ (the irreducible subrepresentations) we can construct ρ straightforward.

In LCFT we encounter a different situation. The representations of the Virasoro algebra are still reducible, so there are still irreducible building blocks. However, the representations are no longer decomposable and we don't know how to use the building blocks to construct the whole representation. This fact makes the analysis of the representation theory of an LCFT much harder than in the case of a conventional CFT.

There is one additional observation we can make if we go back to the full solution of the differential equation (33). As we have seen, it is impossible to get rid of the logarithmic divergence and the full solution of the differential equation always contains this type of divergence. This fact forced us to modify the OPE and the modification led to a new kind of operator with unusual properties. As we learned in this section, the new properties are responsible for off-diagonal terms in the representation of L_0 which can't be transformed away. A similarity transformation brings L_0 into Jordan block form and this finally implies reducible but indecomposable representations of the Virasoro algebra. These lines of reasoning do not only show a nice interplay between calculus and algebra, but it turns out that the logic can be inverted and starting from algebraic properties of the theory, one can derive the analytic properties[4]. This opens the door to two (equivalent) definitions of LCFT:

- *Analytic definition*: A logarithmic CFT is a two-dimensional CFT where some of the correlators show a logarithmic divergence.
- *Algebraic definition*: A logarithmic CFT is a two-dimensional CFT where the representation of the Virasoro algebra is reducible but indecomposable.

The first definition is the one we implicitly used through out this report, but the second one represents the modern point of view.

3.2 The full correlator

Since in conventional CFT holomorphic and anti-holomorphic parts often factorize (as for example in $\phi_{h,\bar{h}}(z,\bar{z})) = \phi_h(z)\phi_{\bar{h}}(\bar{z})$), it is convenient to restrict the analysis of a problem to the holomorphic part and construct the full solution *a posteriori* by multiplying with the appropriate anti-holomorphic part. In the case of the full correlator of a proper CFT this factorization reads[1]

$$\sum_k G_k(z)G_k(\bar{z}) \tag{47}$$

where $G_k(z)$ denotes the k-th holomorphic solution of a differential equation and $G_k(\bar{z})$ is the corresponding anti-holomorphic solution. The above procedure is consistent with the requirement that the correlator should be a single-valued analytic function. However, one can show that in the logarithmic case, the requirement of a single-valued correlator leads to the expression[1] (for the special case discussed in section 2 of this report)

$$G_1(z)G_2(\bar{z}) + G_1(\bar{z})G_2(z). \tag{48}$$

This "non-diagonal" mixing of holomorphic and anti-holomorphic parts is basically due to the appearance of the logarithm which is ambiguous for complex variables.

The above consideration shows that in LCFT the full solution of a problem, *i.e* the solution containing holomorphic and anti-holomorphic parts, is not always trivial to find. A little more care is required.

3.3 Null vectors

We have seen that null vectors appear quite naturally in the framework of conventional CFT. Mathematically speaking, null vectors are just linear combinations of finitely many vectors from the same Verma module, which turn out to be linearly dependent and thus decouple from all other states in the Hilbert space. So, null vectors are simply part of the mathematical structure of the conventional CFT. From a physical point of view, null vectors are a manifestation of local conformal invariance.
We discussed this point of view in some detail and we recognized its importance. In fact we learned that null vectors are a powerful tools when it comes to calculate correlators, which involve more than three fields. Due to their computational power, we would like to have null vectors also in the logarithmic CFT. We know of course, that there are null vectors in LCFT! A big part of this report relys on the existence of a null vector at second level for the specific $c = -2$ CFT we studied in section 2. However, this is a null vector which lives in the Verma module $\mathcal{V}_{h,c}^{(2)}$ of the primary field μ. Since any LCFT contains besides some primary fields also their respective logarithmic partners, we are tempted to ask, if there are null vectors for the logarithmic operators. The answer is yes, but they are harder to find.
As first attempt to find a logarithmic null vector one could try the following procedure: Suppose $|\chi_\phi\rangle$ is a null vector in the Verma module of the primary field ϕ. Then, the replacement $\phi \to \psi$ (where ψ is the logarithmic partner of ϕ) should lead to a new null vector (due to the observation, that both operators have the same conformal dimension). This procedure clearly works for primary fields of the same conformal dimension, but it can't be correct for logarithmic operators.
The reason lies in the unusual properties of the LCFT. It is also true for more general LCFTs than the one we have studied here, that the action of the Virasoro modes on HWS produces off-diagonal terms. These terms mix contributions coming from primary fields and logarithmic fields and leads to two consequences: The procedure described above doesn't work and the structure of the Verma module of a logarithmic field is more complicated than the conventional one. From the failure of this naïve procedure we learn, that we have to study the structure of the Verma module in more detail. The insights we gain by this analysis shed light on the right path towards

a construction of logarithmic null vectors. However, this kind of analysis is beyond the scope of this report and we therefore just mention that it is indeed possible to construct a formalism to find logarithmic null vectors, which is similar to the formalism developed in Stephanie Mayer's talk. More details can be found in reference [2].

3.4 Transformation properties of the logarithmic field

In some of the preceeding talks and also in this report, it was important to compute the action of a Virasoro mode on a field within a correlator. We know exactly what to do when we work with primary fields only. As soon as we consider also logarithmic operators, we have to account for the unusual properties of those operators and according to [2] we find

$$L_n\langle\phi_{h_1}(z_1)\cdots\phi_{h_k}(z_k)\rangle = \sum_{i=1}^{k} z^n\left[z\partial_i + (n+1)(h_i+\Delta_{h_i})\right]\langle\phi_{h_1}(z_1)\cdots\phi_{h_k}(z_k)\rangle \tag{49}$$

where the nilpotent operator Δ_{h_i} is a manifestation of the off-diagonal contributions. As Flohr explains further in reference [2], equation (49) is a reflection of the transformation properties of a logarithmic CFT under conformal transformations of the fields. It can be shown that the transformed field is given by

$$\phi_{h_i}(z) = \left(\frac{\partial f(z)}{\partial z}\right)^{h_i}\left[1+\log(\partial_z f(z))\Delta_{h_i}\right]\phi_{h_i}(f(z)). \tag{50}$$

At this point we have to be very careful. Since we consider a logarithmic CFT we could have correlators containing only primary fields, only logarithmic fields or a combination of both fields. This means that in (49) the Virasoro mode could act on both types of fields. Secretely, we introduced a new notation to take this into account. The field ϕ_{h_i} could be both, a primary field Φ_{h_i} or the corresponding logarithmic partner Ψ_{h_i}. Moreover, the operator Δ_{h_i} doesn't only account for the off-diagonal contributions of logarithmic fields but it allows us to treat different types of correlators at the same time, just by virtue of its properties: $\Delta_{h_i}\Phi_{h_j}=0$ and $\Delta_{h_i}\Psi_{h_j}=\delta_{ij}\Phi_{h_i}$. We see, that if a correlator contains only primary fields, we recover from (49) the result of conventional CFT. If there are also logarithmic fields contained in the correlator, the Δ_{h_i} operator produces additional terms which do not appear in conventional CFT.
The properties of this nilpotent operator affect also the transformation law

(50). Again, if $\phi_{h_i} = \Phi_{h_i}$ we recover the usual transformation law for a primary field. But if we subject a logarithmic field to a conformal transformation $z \to f(z)$, it behaves quite differently than a primary field. There is an additional term which is proportional to $\log(\partial_z f(z))\Phi_{h_i}(f(z))$.

3.5 Modified Ward identities

In section 1 we haven seen how the conformal Ward identities can be derived by demanding global conformal invariance. After the discussion of the transformation properties of logarithmic fields it should be clear, that the conformal Ward identities have to be modified in the context of a logarithmic CFT. To what extend they have to be modified can be seen as follows: Starting from (50) one can derive, using the same techniques as in conventional CFT, an analogue of equation (1). Since also in the logarithmic case it holds true that only L_{-1}, L_0 and L_1 are the only Virasoro modes which are compatible with global conformal invariance, we can use the same procedure to derive the conformal Ward identities. Putting all the steps together one arrives at a modifies system of three partial differential equations for the N-point function G_N:

$$\begin{aligned} L_{-1} \quad &: \qquad \sum_{i=1}^{N} \partial_{z_i} G_N(z_1, ..., z_N) = 0 \\ L_0 \quad &: \qquad \sum_{i=1}^{N} (z_i \partial_{z_i} + (h_i + \Delta_{h_i})) G_N(z_1, ..., z_N) = 0 \\ L_1 \quad &: \qquad \sum_{i=1}^{N} (z_i^2 \partial_{z_i} + 2z_i(h_i + \Delta_{h_i})) G_N(z_1, ..., z_N) = 0. \end{aligned} \tag{51}$$

These differential equations look nearly the same as in the case of a conventional CFT. The only difference lies in the appearance of the nilpotent operator Δ_{h_i}. In the case of a corelator which contains only primary fields, the action of Δ_{h_i} vanishes and we recover the conformal Ward identities of conventional CFT. This is a sign of consistency: The form of the four-point function in section 2 follows from the conventional Ward identities and from the detailed analysis of section 2 we constructed an extended theory. Thus, the conventional Ward identities have to be contained in the extended theory as a special case because any deviation would lead to a different form of the four-point function. This would contradict our starting point of the analysis and the theory wouldn't be self-consistent.
A more intresting case is provided by a correlator which contains logarithmic

fields or a combination of both field types. Whenever the Δ_{h_i} operator hits a logarithmic field, it creates a primary field and an additional term (which is absent in the conventional case) appears. Clearly, this additional terms and the mixing of primary and logarithmic contributions (especially in the case of a correlator with a pure logarithmic content) affect the solutions of the equations. As an example, one could try to compute the two-point function $\langle \Psi_{h_1}(z_1)\Psi_{h_2}(z_2)\rangle$ of two logarithmic operators:

$$\begin{aligned}
&\sum_{i=1}^{2} \partial_{z_i} \langle \Psi_{h_1}\Psi_{h_2}\rangle = 0 \\
&\sum_{i=1}^{2} (z_i\partial_{z_i} + h_i)\langle \Psi_{h_1}\Psi_{h_2}\rangle + \langle \Phi_{h_1}\Psi_{h_2}\rangle + \langle \Psi_{h_1}\Phi_{h_2}\rangle = 0 \\
&\sum_{i=1}^{2} (z_i^2\partial_{z_i} + 2z_i h_i)\langle \Psi_{h_1}\Psi_{h_2}\rangle + \langle \Phi_{h_1}\Psi_{h_2}\rangle + \langle \Psi_{h_1}\Phi_{h_2}\rangle = 0.
\end{aligned} \tag{52}$$

The differential equations are more difficult than in the conventional case (and look even worse when one reinserts the z_i-dependence), but they are still solvable for this special case. The solution is found to be

$$\langle \Psi_{h_1}(z_1)\Psi_{h_2}(z_2)\rangle = \delta_{h_1,h_2} \frac{B - 2A\log(z_1 - z_2)}{(z_1 - z_2)^{2h_1}} \tag{53}$$

where A and B are two constants and the form resembels the one of the two-point funcion of two primary field in the conventional case:

$$\langle \Phi_{h_1}(z_1)\Phi_{h_2}(z_2)\rangle = \delta_{h_1,h_2} \frac{C}{(z_1 - z_2)^{2h_1}}. \tag{54}$$

However, the logarithm is a new feature and it clearly changes the behavior of the two-point function.

4 Applications of LCFT

So far, we understand LCFT as an extension of conventional CFT. Logarithmic divergencies forced us to introduce logarithmic operators and LCFT can thus be regarded as a richer CFT. But is this "richness" just a mathematical curiosity or do LCFTs appear in actual applications?
Indeed, already Gurarie discussed possible applications of LCFT in his original 1993 paper. He shows that the free ghost model described by the action

$$\mathcal{S} \sim \int \mathrm{d}^2 z \partial\theta\bar{\partial}\bar{\theta} \tag{55}$$

where θ is a Grassmann variable, gives rise to a logarithmic conformal field theory. As he explains further, this type of model is used in polymer physics (see [1] and reference therein).
An other early application of LCFT is percolation. Percolation is related to a generalization of the Ising model known as Potts model and can be understood as a critical two-dimensional system, which is studied in statistical physics. A good introduction to the topic is provided by J. Cardy in [5]. More on critical two-dimensional systems, percolation and their relation to LCFT can be found in an original work of V. Gurarie and A. Ludwig in [6]. In reference [7] by M. Flohr, several applications of LCFT are mentioned concretely (but note that the paper is written in German). Flohr briefly discusses conformal turbulences in two-dimensional systems, Seiberg-Witten theory and the fractional quantum hall effect. It is also mentioned, that the quantum hall effect with filling factor $\nu = 5/2$ can only be explained by a $c = -2$ LCFT.
Clearly, logarithmic conformal field theory is not a mathematical curiosity. It extends the framework of conventional CFT and leads to a new regime of applications.

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