Modular invariance and orbifolds

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Outline

Topics

- The modular group $SL_2(\mathbb{Z})$
  - Modular transformations
  - Generators of the modular group
  - Special functions and their modular properties
- Conformal field theory on a torus
  - The free boson and fermion
  - Variation: Compactified boson
- Orbifolds
  - The $\mathbb{Z}_2$ orbifold theory for compactified bosons

Modular transformations

Definition

The modular group Γ is the group of all linear fractional transformations of the upper half complex plane $\mathbb{H}$ of the form

$$z \mapsto \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

Modular transformations

Group properties

- Identify (a, b, c, d) transformation with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
- Identity: $a = 1, b = 0, c = 0, d = 1$ corresponds to $\mathbb{1}$.
- Composition corresponds to matrix product and is associative.
- Inverse to (a, b, c, d): $(d, -b, -c, a)$ - like matrix inverse.
- No difference between the transformation (a, b, c, d) and $(-a, -b, -c, -d)$.

Modular transformations

Matrix group

- As $ad - bc = 1$, the matrices of modular transformations have unit determinant.
- $\Rightarrow \mathrm{SL}_2(\mathbb{Z})$, the special linear group.
- Matrices only determined up to a sign!
- $\Gamma \cong \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) / \{\mathbb{1}, -\mathbb{1}\}$.
- "Projective special linear group". From now on, write $\mathrm{SL}_2(\mathbb{Z})$.

Generators of the modular group

$\mathcal{T} and \mathcal{S}$

Define:

- $\mathcal{T} : \mathbb{H} \to \mathbb{H}, z \mapsto z + 1$. $\mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
- $\mathcal{S} : \mathbb{H} \to \mathbb{H}, z \mapsto -\frac{1}{z}$. $\mathcal{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Defining properties

It is also possible to arrive at $\mathcal{T}$ and $\mathcal{S}$ via their defining properties:

$$(\mathcal{S}\mathcal{T})^3 = \mathcal{S}^2 = \mathbb{1}.$$

Theta functions

Origin

- Holomorphic functions of $(z, \tau) \in \mathbb{C} \times \mathbb{H}$.
- Important for the theory of elliptic functions.
- Arise as solutions of the heat equation.
- Connected to Riemann's ζ function via an integral transformation.

Theta functions

Definition (z = 0)

Let $\tau \in \mathbb{H}$, let $q = \exp(2\pi i \tau)$.

- $\Theta_2(\tau) = \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2/2} = 2q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1-q^n)(1+q^n)^2$.
- $\Theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} = \prod_{n=1}^{\infty} (1-q^n)\left(1+q^{n-\frac{1}{2}}\right)^2$.
- $\Theta_4(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} = \prod_{n=1}^{\infty} (1-q^n)\left(1-q^{n-\frac{1}{2}}\right)^2$.

Dedekind's η function

Definition

$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$.

Connection to theta functions

$\eta^3(\tau) = \frac{1}{2}\Theta_2(\tau)\Theta_3(\tau)\Theta_4(\tau)$.

Modular properties

Table of modular properties

$\eta(\tau+1) = \exp\left(\frac{\pi i}{12}\right)\eta(\tau)$	$\eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\eta(\tau)$
$\Theta_2(\tau+1) = \exp\left(\frac{\pi i}{4}\right)\Theta_2(\tau)$	$\Theta_2\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\Theta_4(\tau)$
$\Theta_3(\tau+1) = \Theta_4(\tau)$	$\Theta_3\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\Theta_3(\tau)$
$\Theta_4(\tau+1) = \Theta_3(\tau)$	$\Theta_4\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\Theta_2(\tau)$

The torus

Definition

- Riemann genus: 1
- A parallelogram whose opposite edges are identified.
- The torus has two periods ω_1, ω_2. Points which differ by integer combinations of ω_1, ω_2 are identified.
- The quantity of interest is the modular parameter $\tau = \frac{\omega_2}{\omega_1}, \tau \in \mathbb{H}$.

The torus

Modular transformations

- $\tau \in \mathbb{H} \Rightarrow \mathrm{SL}_2(\mathbb{Z})$ can act on τ.
- $\mathcal{S}$: Looking at the torus from the side.
- $\mathcal{T}$: Cutting the torus, rotating one piece by 2π, stick back together.
- Modular transformations of τ do not change the torus.

The partition function

Establishment

- Define space and time directions along real and imaginary axes.
- Translation operator over distance a, parallel to ω_2 in space-time: $\exp\left(-\frac{a}{|\omega_2|}\left[H\ \mathrm{Im}\omega_2 - iP\ \mathrm{Re}\omega_2\right]\right)$.
- Regard a as lattice spacing. Complete period contains m lattice spacings ($|\omega_2| = ma$), then $Z(\omega_1, \omega_2) = \mathrm{Tr}\exp\left(-\left[H\ \mathrm{Im}\omega_2 - iP\ \mathrm{Re}\omega_2\right]\right)$.

The partition function

In terms of Virasoro generators

- Regard the torus as a cylinder of circumference L whose ends have been stuck together.
- Then $H = \frac{2\pi}{L}\left(L_0 + \overline{L}_0 - \frac{c}{12}\right)$, $P = \frac{2\pi i}{L}\left(L_0 - \overline{L}_0\right)$.
- $\Rightarrow Z(\tau) = \mathrm{Tr}\left(q^{L_0 - \frac{c}{24}}\overline{q}^{\overline{L}_0 - \frac{c}{24}}\right)$.

The free boson on the torus

Partition function

- Remember $\chi_{(c,h)}(\tau) = \mathrm{Tr} q^{L_0 - \frac{c}{24}} = \frac{q^{h + \frac{1-c}{24}}}{\eta(\tau)}$.
- $\Rightarrow Z_{bos} \propto \frac{1}{|\eta(\tau)|^2}$.
- Not modular invariant!
- $Z_{bos}(\tau) = \frac{1}{\sqrt{\mathrm{Im}\tau}|\eta(\tau)|^2}$ is modular invariant.

The free boson on the torus

Detailed derivation with ζ regularization

- Path-integral formulation.
- Result is a divergent product of the form $\prod_n \left(\frac{1}{\lambda_n}\right)^{\frac{1}{2}}$.
- Define a ζ-like function $G(s) = \sum_n \frac{1}{\lambda_n^s}$.
- After analytic continuation, our product is regularized to be $\exp\left(\frac{1}{2}G'(0)\right)$.

The free fermion on the torus

Action

- Free-fermion action: $S = \frac{1}{2\pi}\int d^2x \ \left(\overline{\psi}\partial\overline{\psi} + \psi\bar{\partial}\psi\right)$.
- $\psi, \overline{\psi}$ are decoupled.
- $\Rightarrow Z = \mathrm{Pf}(\partial)\mathrm{Pf}(\bar{\partial}) = \sqrt{\det \nabla^2}$.

The free fermion on the torus

Periodicity conditions

- $\psi(z+\omega_1) = e^{2\pi i v}\psi(z), \quad \psi(z+\omega_2) = e^{2\pi i u}\psi(z).$
- Action must be invariant when $z \mapsto z+\omega_1$ or $z \mapsto z+\omega_2$.

Possible periodicity conditions

$(v,u) = (0,0)$	(R,R)
$(v,u) = (0,\frac{1}{2})$	(R,NS)
$(v,u) = (\frac{1}{2},0)$	(NS,R)
$(v,u) = (\frac{1}{2},\frac{1}{2})$	(NS,NS)

R: Ramond, NS: Neveu-Schwarz.

The free fermion on the torus

Periodicity conditions

- A set (v, u) of periodicity conditions is called a spin structure.
- Decoupled ψ, $\overline{\psi}$: consider partition function obtained by integrating the holomorphic field only, $d_{v,u}$.
- $\Rightarrow Z_{v,u} = |d_{v,u}|^2$.
- When implementing the conditions, find operator anticommuting with $\psi(z)$:

$$(-1)^F, \quad F = \sum_{k \geq 0} F_k, \quad F_k = b_{-k} b_k.$$

The free fermion on the torus

Associated partition functions

$$d_{0,0} = 0,$$

$$d_{0,\frac{1}{2}} = \sqrt{\frac{\Theta_2(\tau)}{\eta(\tau)}},$$

$$d_{\frac{1}{2},0} = \sqrt{\frac{\Theta_4(\tau)}{\eta(\tau)}},$$

$$d_{\frac{1}{2},\frac{1}{2}} = \sqrt{\frac{\Theta_3(\tau)}{\eta(\tau)}}.$$

The free fermion on the torus

Modular invariance

- Check modular properties of $d_{0,\frac{1}{2}}, d_{\frac{1}{2},0}, d_{\frac{1}{2},\frac{1}{2}}$.
- Up to phase factors, they mix.
- $\Rightarrow$ All the three possibilities (NS,R), (R,NS), (NS,NS) have to be included in the theory.

$$\begin{aligned} Z &= Z_{\frac{1}{2},\frac{1}{2}} + Z_{0,\frac{1}{2}} + Z_{\frac{1}{2},0} \\ &= \left|\frac{\Theta_2}{\eta}\right| + \left|\frac{\Theta_3}{\eta}\right| + \left|\frac{\Theta_4}{\eta}\right| \\ &= 2\left(|\chi_{1,1}|^2 + |\chi_{2,1}|^2 + |\chi_{1,2}|^2\right) \end{aligned}$$

- This is twice the partition function of the Ising model.

The compactified boson

Boundary conditions

- Consider the boundary condition:

$$\varphi(z + k\omega_1 + k'\omega_2) = \varphi(z) + 2\pi R(km + k'm'), \quad k, k' \in \mathbb{Z}.$$

- Integration: Decompose $\varphi = \varphi^{cl}_{m,m'} + \tilde{\varphi}$.
- $Z_{m,m'}(\tau) = Z_{bos}(\tau) \exp\left[-\frac{\pi R^2 |m\tau - m'|^2}{2\mathrm{Im}\,\tau}\right]$.

The compactified boson

Modular invariance

- $\mathcal{S}$ and $\mathcal{T}$ act on $Z_{m,m'}$ as follows:

$$Z_{m,m'}(\tau+1) = Z_{m,m'-m} \quad Z_{m,m'}\left(-\frac{1}{\tau}\right) = Z_{-m',m}.$$

- $\Rightarrow$ Sum over all (m, m') with equal weights.
- The final partition function is

$$Z(R) = \frac{1}{|\eta(\tau)|^2} \sum_{e,m\in\mathbb{Z}} q^{\left(\frac{e}{R}+\frac{mR}{2}\right)^2/2} \overline{q}^{\left(\frac{e}{R}-\frac{mR}{2}\right)^2/2}.$$

The compactified boson

The final partition function

- Sum over all (electric) charges of vertex operators and all possible "winding numbers" (magnetic charges) of the $c = 1$ Virasoro characters squared.
- Conformal dimensions:

$$h_{e,m} = \frac{1}{2}\left(\frac{e}{R} + \frac{mR}{2}\right)^2, \quad \bar{h}_{e,m} = \frac{1}{2}\left(\frac{e}{R} - \frac{mR}{2}\right)^2.$$

- The model has a $e \leftrightarrow m$ duality

$$Z\left(\frac{2}{R}\right) = Z(R).$$

Orbifolds

Definition

Let $\mathcal{M}$ be a manifold with a discrete group action $\mathcal{G} : \mathcal{M} \to \mathcal{M}$. $\mathcal{G}$ possesses a fixed point $x \in \mathcal{M}$ if for $g \in \mathcal{G}, g \neq \mathbb{1}$, we have $gx = x$. Then we construct the orbifold $\mathcal{M}/\mathcal{G}$ by identifying points under the equivalence relation $x \sim gx$ for all $g \in \mathcal{G}$.

Orbifolds

Properties

- Generalization of manifolds - allows discrete singular points.
- If $\mathcal{G}$ acts freely (no fixed points) $\Rightarrow \mathcal{M}/\mathcal{G}$ is a manifold.
- Fixed points lead to singularities.

The $\mathcal{S}_1/\mathbb{Z}_2$ orbifold

Example

- Take $\mathcal{M} = \mathcal{S}_1$, the circle, with $x \equiv x + 2\pi r$.
- Let $\mathcal{G} : \mathbb{Z}_2 : \mathcal{S}_1 \to \mathcal{S}_1$ with the generator $g : x \mapsto -x$.
- Fixed points: $x = 0$, $x = \pi r$.

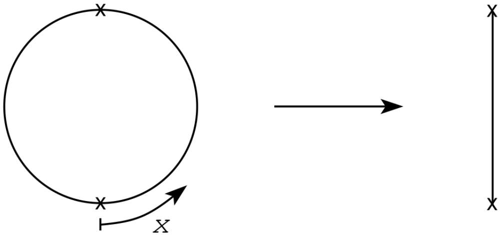

Figure: The $\mathcal{S}_1/\mathbb{Z}_2$ orbifold.

The $\mathcal{S}_1/\mathbb{Z}_2$ orbifold

Application

- In CFT: Take modular invariant theory $\mathcal{T}$ and a discrete symmetry $\mathcal{G}$ on its Hilbert space. Construct a "modded-out" theory $\mathcal{T}/\mathcal{G}$ which is also modular invariant.
- Take the $\mathbb{Z}_2$ action on the compactified free boson.
- We have more general boundary conditions:

$$\varphi(z + k\omega_1 + l\omega_2) = e^{2\pi i(kv+lu)}\varphi(z).$$

- The action is invariant under $\varphi \mapsto -\varphi \Rightarrow$ only half the path-integral range as compared to circle.

The $\mathcal{S}_1/\mathbb{Z}_2$ orbifold

Application

- Calculate holomorphic partition functions like for the free fermion ($Z_{v,u} = |f_{v,u}|^2$):

$$f_{0,\frac{1}{2}} = 2\sqrt{\frac{\eta(\tau)}{\Theta_2(\tau)}},$$

$$f_{\frac{1}{2},0} = 2\sqrt{\frac{\eta(\tau)}{\Theta_4(\tau)}},$$

$$f_{\frac{1}{2},\frac{1}{2}} = 2\sqrt{\frac{\eta(\tau)}{\Theta_3(\tau)}},$$

The $\mathcal{S}_1/\mathbb{Z}_2$ orbifold

The final partition function

$$Z_{orb}(R) = \frac{1}{2}\left(Z(R) + \frac{|\Theta_2\Theta_3|}{|\eta|^2} + \frac{|\Theta_2\Theta_4|}{|\eta|^2} + \frac{|\Theta_3\Theta_4|}{|\eta|^2}\right).$$

Conclusion

Key points

- Prediction of partition functions from invariance considerations.
- ζ regularization of divergent sums/products.
- Modular invariance restricts the theory.
- Construction of orbifold theories.