From SLE to conformal field theory

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This report is a master student level introduction to Schramm-Loewener-Evolution (SLE) and the study of conformally invariant two dimensional pictures using SLE. Connections to conformal field theory will be discussed briefly.

Contents

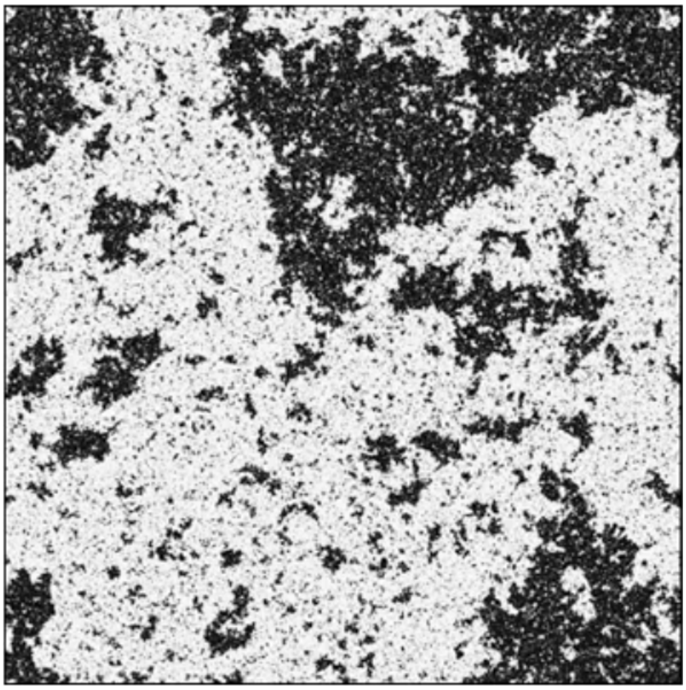

Figure 1: Ising magnets at critical temperature, scale invariance and local (nearest neighbor) interaction between the magnets lead to conformal invariance

1 Introduction

Schramm-Loewener-Evolution (SLE) are a tool to understand two dimensional growth processes. However, in the special case of stochastic Schramm-Loewner-Evolution it can be used to understand two dimensional conformally invariant pictures, for example the picture of Ising magnets at critical temperature (see Figure 1).

One main idea is to express conformal invariance as **scale invariance** plus *local interaction/locality* like in the Ising model where the Hamiltonian usually describes nearest neighbor interaction only. Scale invariance means that we can zoom in or out and the picture still has the same properties. Throughout the report scale invariance and locality will be expressed in other more complicated ways, which is why I will make the corresponding terms bold if it is a synonyme for scale invariance and italic if it is one for locality, to get an easier connection.

Another important idea is a description of the conformally invariant pictures based on the borders of regions of the same color (the same spin). We are going to call these borders interfaces and we will be able to describe the interfaces as paths in a mathematical sense. So our ultimate goal will be to find an equation for these paths, and if possible, the solution.

In this report first we are going to study discrete two dimensional conformally invariant pictures to get an idea about what conformal invariance means and how to express it in

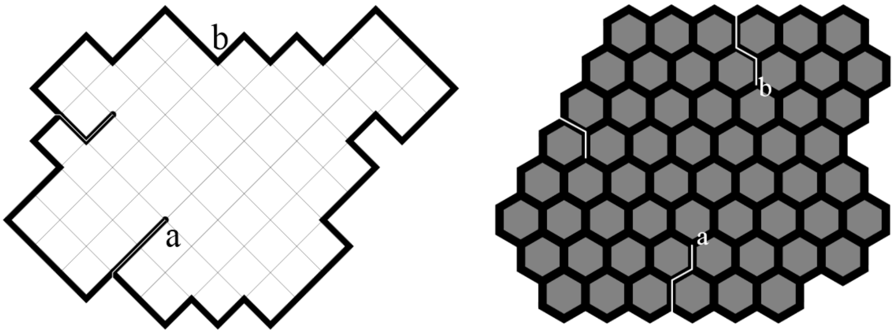

Figure 2: A sqare and a hexagonal lattice domain. Notice that a and b are not in the domain, but in the closure of the domain.

a useful way. In the second chapter we will go to the continuous limit to get some sort of mathematical equation that these conformally invariant pictures have to satisfy. And last we will try to see the connections between SLE and another powerful theory which is able to interpret two dimensional conformally invariant things, that is Conformal Field Theory.

This text is a part of a Proseminar for bachelor and master students, where I held a presentation about the topic April of 2013. So my goal for this report was to make it easy enough so that a student like myself can get the key points of the topic in a few hours. To achieve this goal I will focus on explaining the concepts instead of spending a lot of time deriving the rather complicated underlying math. For those interested in that, consult the items in the bibliography.

2 Discrete growth processes

2.1 Hexagonal lattice domains

Since our objects of interest are two dimensional conformally invariant pictures, everything can be done on the complex plane. A set on the complex plane is a domain, if it is open and simply connected (the picture we keep in mind for simple connectedness is contractibility). A hexagonal or square lattice domain $\mathbb{D}$ is a domain, which can be decomposed as a union of open hexagons or squares with side length 1 (faces), open segments of length 1 (edges) and points (vertices). Note that the edges always need to be between two hexagons or squares, they cannot be part of the boundary of the domain. Likewise the vertices are always in the interior of the domain and not part of the boundary. These are essential conditions for the domain to be open.

Imagine that every face is one spin. So by coloring the faces black and white, we can describe the state the system is in. Since we are interested in a description based on interfaces (that are borders between big regions of the same spin), we try to express those as paths inbetween spins. So the first and easiest example to look at is a domain with one interface (path), that cuts the domain into two pieces, one mainly spin up, one

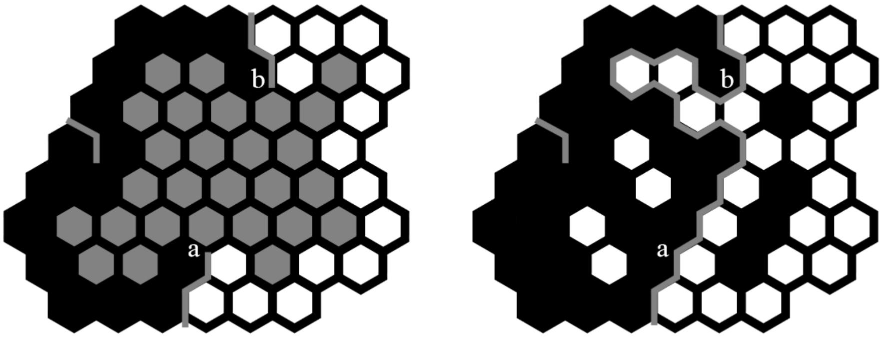

Figure 3: Left: the black hexagons are on the left side of every path from a to b, the white ones are always on the right side. However the grey ones depend can be either.
Right: One specific configuration, every grey hexagon was given a color. In this case the path from a to b is unique.

spin down(there can be small island of opposite spin in the halves). For the resulting theory that will be discussed, this is actually the only example we need.

With this we motivated the following definitions: An admissible boundary condition is a couple of distinct points (a, b), $a, b \notin \mathbb{D}$, such that a and b can be connected by a path $\gamma \in \mathbb{D}$. For illustration look at Figure 3. We see that (a, b) are part of the closure of $\mathbb{D}$. Also γ isn't a path in the usual continuous sense, we can define it as:

A path $\gamma \in \mathbb{D}$ from a to b is a sequence $s_1, ..., s_{2n+1}$, where

- $s_1 = a$ and $s_{2n+1} = b$
- the s_{2m+1} are distinct vertices
- the s_{2m} are distinct edges with boundary $\{s_{2m-1}, s_{2m+1}\}$

Since any such path splits $\mathbb{D}$ into a left and a right piece, we can find faces that are always on the right (left) side of the path from a to b, we call these faces right (left) faces (see Figure 3). There are also faces that are on the right side of some paths and on the left side of others (colored gray in Figure 3), we call these grey or inner hexagons.

One further definition I would like to make is the triple $(\mathbb{D}, a, b)$ as hexagonal lattice domain $\mathbb{D}$ with admissible boundary condition (a, b).

2.2 Examples: Percolation and the Ising model

If we now give white or black color to the grey hexagons in an arbitrary way, the resulting path from a to b is unique. The proof is an induction and the main idea is to start at a and take the edge, which is between a black and a white hexagon, to the next vertice. There is only one such edge. The vertice reached becomes the new a and repeat.

As a stochastic example we can color the grey (inner) hexagons with a fair coin $p = 1/2$, so every spin is an independent random variabale. The induced probability distribution

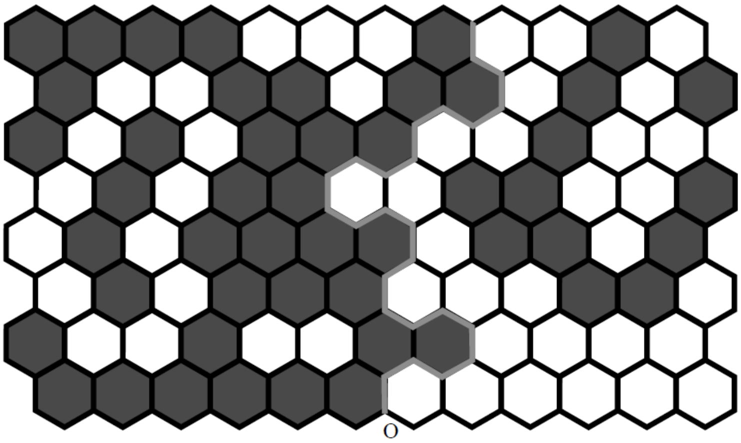

Figure 4: One example of percolation distribution, every inner hexagon was colored with a fair coin.

for the path from a to b is called percolation probability distribution. Not every path is equally probable, in fact, a path touching l disctinct faces has probability 2^{-l}.

By giving color to faces with a fair coin we can construct an iterative percolation growth process. Start with a given boundary (in Figure 5 the boundry is the real axis plus infinity so that the growth process takes place on one half of the Riemann's sphere) and a given starting point a (a=0 in Figure 5). Then we just keep repeating the following two steps:

- take the only path between a white and a black hexagon
- flip a fair coin to color the one grey hexagon that the path encounters

The result is illustrated in Figure 5. If one wants to simulate this process, the stopping condition is the path hitting the boundry in any point but infinity (computers cant reach infinity by counting).

Notice that the interaction between the spins in the percolation model is trivial, and hence local

A good example for a model with non-trivial interaction between the spins is the Ising model. Here the interaction typically has the range of one face, called nearest neighbor interaction. The Hamiltonian for this model looks like this:

$$H(\sigma) = -\sum_{<i,j>} J\sigma_i\sigma_j \qquad P_\beta(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z_\beta}$$

So if we have a domain with an admissible boundary condition $(\mathbb{D}, a, b)$ we can colour all the inner hexagons according to the Ising distribution. Again, there will always be exactly one interface (path from a to b) and a number of islands of the same color/spin.

However, not every picture we get this way is scale invariant, for that the parameters of the system need to be adjusted so that we get a critical system. For example in

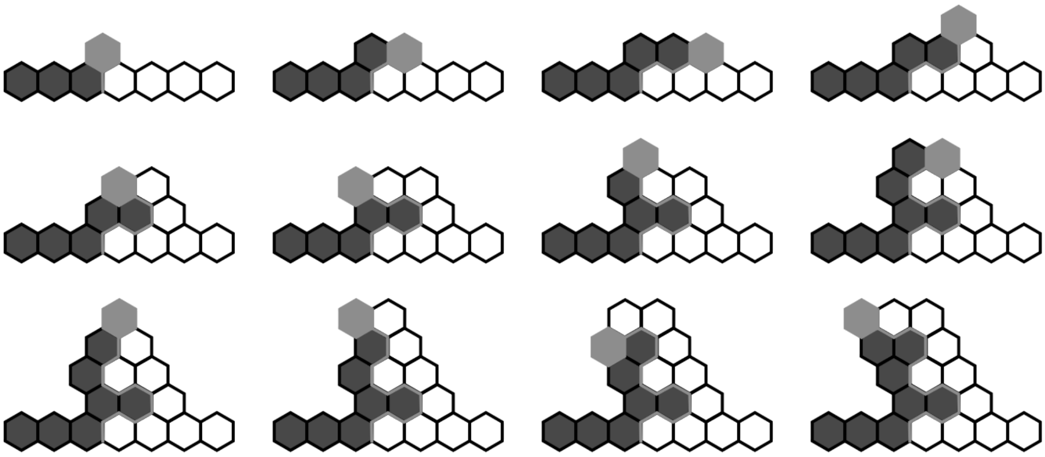

Figure 5: Iterative percolation growth process. For every grey hexagon encountered by the path, a fair coin is used to decide its color

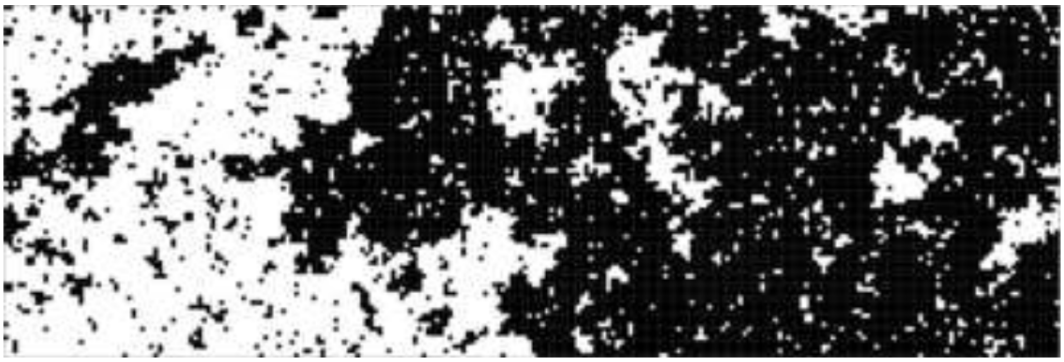

Figure 6: Simulation of an Ising model on a lattice domain with boundary conditions (a, b)

theormodynamics/statistical physics the two dimensional Ising model has two phases, the paramagnetic and the ferromagnetic one. Between those phases there is a phase transition at critical temperature T_C (temperature is practically the inverse of β in the Hamiltonian). At that point and only at that point the system becomes scale invariant (see page 403 in [3]) and we can zoom in and out without changing the properties of the picture and the resulting physics.

So in case of the Ising model at phase transition point we get scale invariant pictures and since the interaction between the spins is local, the resulting picture should be conformally invariant. See Figure 6 for an example of the Ising model.

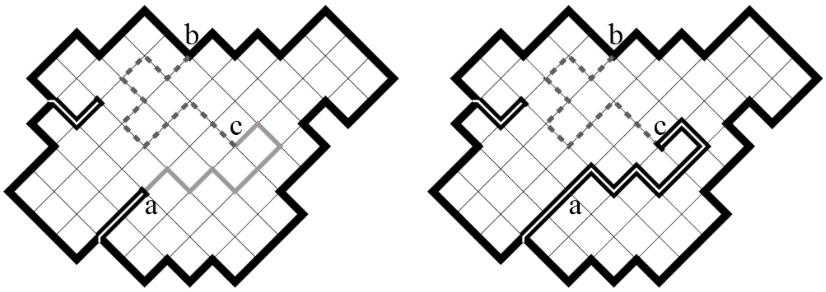

Figure 7: Left: The path from a to c is already fixed, this results in a probability distribution from c to b.
Right: The path from a to c is cut out from the domain, this results in the same probability distribution from c to b like in the fixed case.

2.3 The domain Markov property

One of the two properties we need for conformal invariance is locality. In this chapter we want to find a way to express locality in a different, more useful way, this will be called the *domain Markov property*. But before we need to make a necessary observation.

Given a domain with an admissible boundary condition, $(\mathbb{D}, a, b)$ and $1 \leq m < n$: We can obtain a new hexagonal lattice domain $\mathbb{D}'$ by removing part of the path γ (remove all edges and vertices s_l, $1 < l \leq s_{2m+1}$ from a up to a vertice s_{2m+1} from the domain $\mathbb{D}$ (see Figure 7). Then (s_{2m+1}, b) is an admissible boundary condition for $\mathbb{D}'$, so we get a new triple $(\mathbb{D}', s_{2m+1}, b)$.

With this in mind we want to state a condition every stochastic process with local interaction (like percolation or Ising) should fullfill. For that we want to compare the following probabilities given $(\mathbb{D}, a, b)$:

- the probability distribution for the path $\gamma_{[a,b]}$ in $(\mathbb{D}, a, b)$ given that the path is fixed up to a point c already ($\gamma_{[a,c]}$ fixed)
- the probability distribution of $\gamma_{[c,b]}$ in the domain $\mathbb{D}'$, which we get if we cut out $\gamma_{[a,c]}$ from $\mathbb{D}$

The Domain Markov property is the statement that these two probabilities are equal:

$$P_{(\mathbb{D},a,b)}(.|\gamma_{[a,c]}) = P_{(\mathbb{D}\setminus\gamma_{[a,c]},c,b)}(.)$$

Again, the domain Markov property is out way to express locality in a different, more useful way. In a nutshell it says that for the path to come it doesn't matter what happened further back. If the interaction between the spins would be long range, this would be different.

3 Loewner Chains and stochastic Schramm-Loewner evolution

3.1 Conformally invariant interfaces

Now we want to switch to the continuous limit to find a better, mathematical description for the interfaces. But first, what does continuous limit mean? We still have our domain $\mathbb{D}$ and the boundary conditions (a, b), where the points a and b are still on the closure of the open set $\mathbb{D}$. However, since the lattice spacing goes to zero in the continuous case, there are now infinitely many faces in the domain $\mathbb{D}$, or in other words, every single point in $\mathbb{D}$ is now a face. However, paths, like in the discrete case, still do not contribute. In the discrete case this was because the paths were inbetween the faces, which carry the spin. Now in the continuous case this is because a path has measure zero in 2 dimendions, so if we integrate over an area to get the spin, a path does not contribute, does not make a difference for the total spin. A contributing term would look something like $\rho(x, y)dxdy$, where $\rho(x, y)$ is some kind of spin density, but this will not be important for the following chapters, since we focus on the interfaces, not on the areas of same spin.

We also need to remember one more theorem from complex analysis, namely that two domains $\mathbb{D}$ and $\mathbb{D}'$ are always coformally invariant, i.e. there exists an invertible holomorphic map between them. As a good example to keep in mind we can have a look at Moebius transformations.

$$f : \mathbb{D} \to \mathbb{H} \qquad f(z) = \frac{az + b}{cz + d} \quad a, b, c, d \in \mathbb{R}; ad - bc = 1$$

Moebius transformations have 3 degrees of freedom, since they have four constants but one equation connecting those four constants. Usually what you do is to fix those constants by determining on what points the points $0, z, \infty$ are mapped onto, where x is different from 0 and ∞.

So in the continuous case we still have our domain $\mathbb{D}$ with boundary condition points a,b on its closure and a path $\gamma_{[a,b]}$ connecting them, where the probability distribution for that gamma depends on the interaction law for the spins.

$$\gamma(0) = a \qquad \gamma(\infty) = b \qquad \gamma_{]a,b[} = \gamma(]0, \infty[) \subset \mathbb{D}$$

So that $P_{(\mathbb{D},a,b)}$ is the probability distribution for path $\gamma(]0, \infty[)$ in $\mathbb{D}$.

Since the interation is local, we hope that the domain Markov property still holds in the continuous case.

$$P_{(\mathbb{D},a,b)}(.|\gamma_{[a,c]}) = P_{(\mathbb{D}\setminus\gamma_{[a,c[},c,b)}(.)$$

In a matematical sense this is not a problem, because a domain with a path cut out like that is still a domain, just like in the discrete setting. So we just force this property into our theory and we will see if there exists a solution. We want the domain Markov property to hold.

Now the second thing needed for conformal invariance is scale invariance. Imagine if we map the domain $\mathbb{D}$ with boundary conditions (a, b) onto a, say 10 times bigger copy of itself. Because of the conformal mapping theorem we can do this. The probability

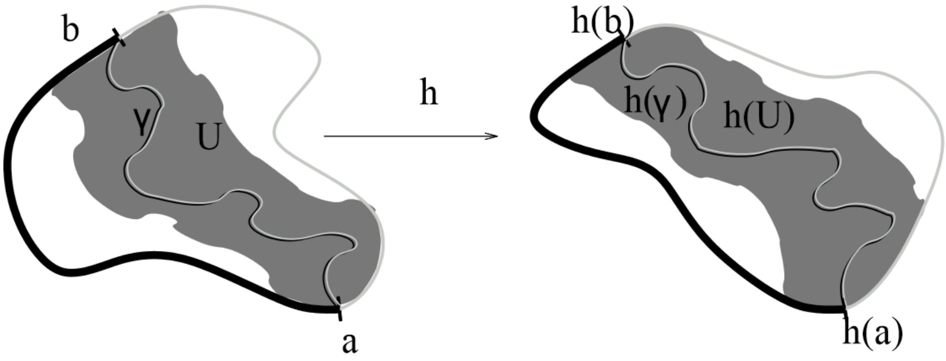

Figure 8: Conformal transport from one domain to another with function h. A reformulation of scale invariance.

distribution for the path in this bigger domain would still be the same, we could give our domain $\mathbb{D}$ an arbitrary size and the properties would not change, scale becomes meaningless for the resulting physics, hence scale invariance. There is a more general expression for what we just have done. It is our way to express scale invariance for the interfaces and it is only based on the knowledge that a map between two domains is always conformally invariant, it is called **Conformal transport** (see Figure 8:

Given a conformal map $h : \mathbb{D} \to \mathbb{D}'$ and given a subset $U \subset \mathbb{D}$ which contains a, the probability distribution for the path $\gamma_{[a,b]}$ in U is the same as the probability distribution for the path $\gamma_{[h(a),h(b)]}$ in $h(U)$:

$$P_{(\mathbb{D},a,b)}(\gamma_{[a,b]} \subset U) = P_{(h(\mathbb{D}),h(a),h(b))}(\gamma_{[h(a),h(b)]} \subset h(U))$$

Now we are able to express both locality of interaction and scale invariance in a, for the interface description, more useful way, namely the domain Markov property and the behaviour of the curves under conformal transformations. What happens now if we combine these two properties in a clever way, what does it mean for our interface?

Start again with a domain and boundary conditions $(\mathbb{D}, a, b)$ and add a point $c \in \mathbb{D}$, so that γ goes through c. Next we fix the path $\gamma_{[a,c]}$, the rest $\gamma_{[c,b]}$ still depends on some probability distribution. Because of the domain Markov property, we know we can cut out $\gamma_{[a,c]}$ without changing this probability distribution of the rest of the path. So we get a new domain $\mathbb{D}' = \mathbb{D}\backslash\gamma_{[a,c]}$ and a new set of boundary conditions (c, b). Now map $\mathbb{D}'$ back onto the original domain $\mathbb{D}$ with the function $h_{\gamma_{[a,c]}} : \mathbb{D}\backslash\gamma_{[a,c[} \to \mathbb{D}$ so that:

$$h_{\gamma_{[a,c]}}(\mathbb{D}\backslash\gamma_{[a,c[}) = \mathbb{D} \quad h_{\gamma_{[a,c]}}(c) = a \quad h_{\gamma_{[a,c]}}(b) = b$$

We know this works because we have seen conformal transport, also we can get a resulting probability distribution for the paths in $\mathbb{D}$ and $\mathbb{D}'$ by putting everything together.

$$P_{(\mathbb{D},a,b)}(\gamma_{[c,b]} \subset U|\gamma_{[a,c]}) = P_{(\mathbb{D},a,b)}(\gamma_{[a,b]} \subset h_{\gamma_{[a,c]}}(U))$$

So we get one equation that expresses conformal invariance for interfaces. The key properties of the probability distribution for $\gamma_{[c,b]}$ are

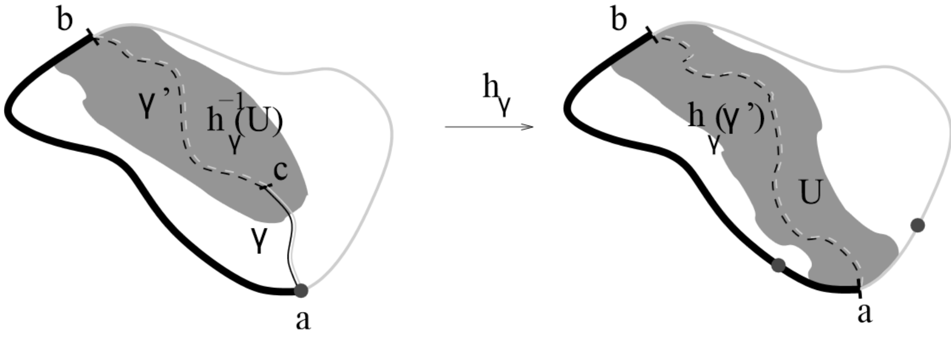

Figure 9: Conformal invariance for paths. The result of combining the equations for locality and scale invariance.

- it is independent of $\gamma_{[a,c]}$ (Markov)
- the same distribution as for $\gamma_{[a,b]}$ (**stationarity of increments**)

Our next goal is to find some sort of differential equation for this mapping back of domains with a path cut out onto itself, so that we can put the aquired properties to use. They will help us find the solution of the differential equation much like symmetries help you in physics to simplify and solve certain problems.

3.2 Loewner Chains

From now on we work on the upper half plane $\mathbb{D} = \mathbb{H}$ with the boundary $\mathbb{R} \cup \infty$, this is called the chordal case. The starting point of the path is set to $\gamma(0) = a = 0$. Fix a path $\gamma_{[a,c]}$ up to some point $c \in \mathbb{H}$. We know that we can map back the domain $\mathbb{H} \backslash \gamma_{[a,c]}$ onto $\mathbb{H}$, but now we want to find some kind of mathematical equation for this mapping back process. Before going on I will say that this equation will not be derives here, it will just be motivated. The derivation involves not only mapping back the domain without a path onto itself but mapping back the domain without general hulls onto itself. Then at the end of the derivation one sets the support for such hulls to a delta function to get the solution for a simple path. But since we want to study simple curves a motivation will satisfy.

First we introduce a "time" parameter t for paths γ_t, which will be associated to the growth of the path. It is important to mention that this is not a physical time, this is just a parameter to describe conformally invariant pictures. If you have a conformal invariant system, let's say an Ising magnet at critical temperature, time becomes irrelevant, the properties of the system stay the same. Now with the help of this parameter t we can chain conformal maps in the following way (see Figure 11):

$$g_t : \mathbb{H} \backslash \gamma_t \to \mathbb{H} \qquad g_s : \mathbb{H} \backslash \gamma_s \to \mathbb{H}$$

$$g_{t+s} : \mathbb{H} \backslash \gamma_{t+s} \to \mathbb{H}$$

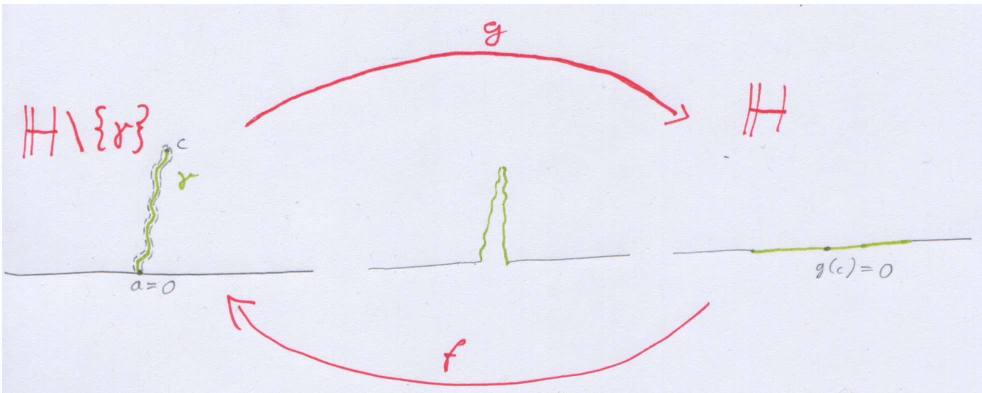

Figure 10: g maps the domain with a path cut out back onto itself, f is the inverse of g. Also notice where the boundary points from the path go, when projexted under g.

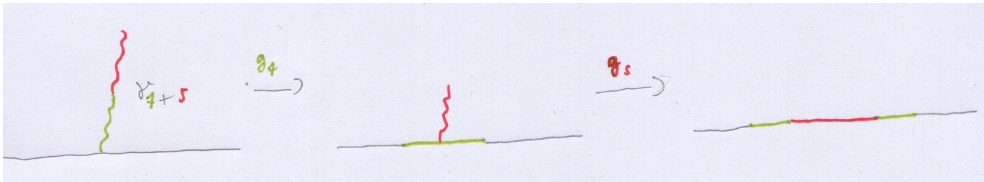

Figure 11: Chaining maps is easy to express with the help of the "time" parameter

As a side note, notice that the function h from the last chapter is different from the g in this chapter. The difference is that h maps back the point c to the point a as defined in the last chapter, while g does not map back the top of γ_t to 0, but just to a point on the real axis. Actually the distance between zero and this point will be an important variable.

We are looking for a differential equation so understanding local growth will be key. So we need a way to compare a γ_t with a curve $\gamma_{t+\epsilon}$, which is just an infinitessimal bit longer.

$$g_t : \mathbb{H} \backslash \gamma_t \to \mathbb{H} \qquad g_{t+\epsilon} : \mathbb{H} \backslash \gamma_{t+\epsilon} \to \mathbb{H}$$

The natural way to do this is just writing down a derivative:

$$\frac{dg_t}{dt} = lim_{\epsilon \to 0} \frac{g_{t+\epsilon} - g_t}{\epsilon}$$

For a simple path one can derive that this differential equation simplifies to:

$$\frac{dg_t}{dt}(z) = \frac{2}{g_t(z) - \xi_t}$$

This actually depends on the parametrization of the path, but we are mostly going to ignore this technicality. The solution $g_t(z)$ to this equation for a given ξ_t and initial condition $g_0(z) = z$ is called Loewner evolution.

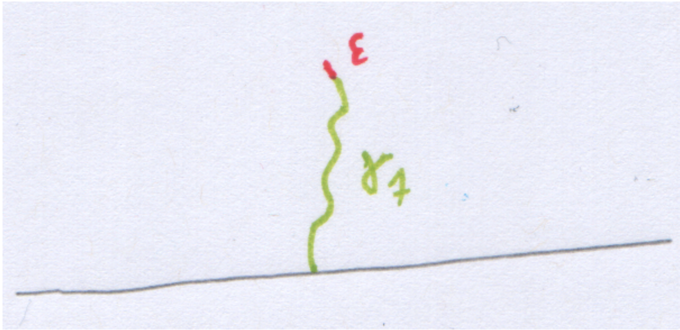

Figure 12: An infinitessimal epsilon piece of path helps expressing the problem as a differential equation.

We now want to understand what this ξ_t does. First of all, this ξ_t is exactly the distance between zero and the point on the real axis which is the image of the top of the path γ_t under the map g_t. The other way around this also means that the image of ξ_t under g_t^{-1} is the top of the path γ_t at time t. Or expressed a little bit more carefully:

$$\gamma_t = lim_{\epsilon \to 0} g_t^{-1}(\xi_t + i\epsilon)$$

But this is not enough to understand what the ξ_t actually does, to illustrate this better we look at the really easy example of the straight line path from 0 to a along the imaginary axis. We parametrize this with the function $g_t = \sqrt{t^2 + a^2}$. Putting this into our differential equation results in

$$\frac{dg_t}{dt}(z) = \frac{2}{g_t(z)}$$

, which means that for the case of the straight line $\xi_t = 0$. So what ξ_t really does is to make the path not go in a straight line, all curvature is caused by ξ_t, from now on we will call ξ_t the driving term. In the following chapter we revisit curves which depend on a probability distribution. So with the knowledge we now have we can already say that this directly translates to a ξ_t based on a probability distribution.

3.3 Chordal SLE

Since the only thing that can prevent the path from going in a straight line is the driving term xi_t, all probabilistic behavior has to be hidden in this term. So from now on xi_t is a random variable which expresses how much the path deviates from a straight line and our goal is to find a solution to the Loewner Chain differential equation

$$\frac{dg_t}{dt}(z) = \frac{2}{g_t(z) - \xi_t}$$

by using the properties we have already derived for conformally invariant interfaces (Markov, stationarity of increments). But first, since in chapter 3.2 c was always projected back onto a, we need to change our function so that the top of the curve is always

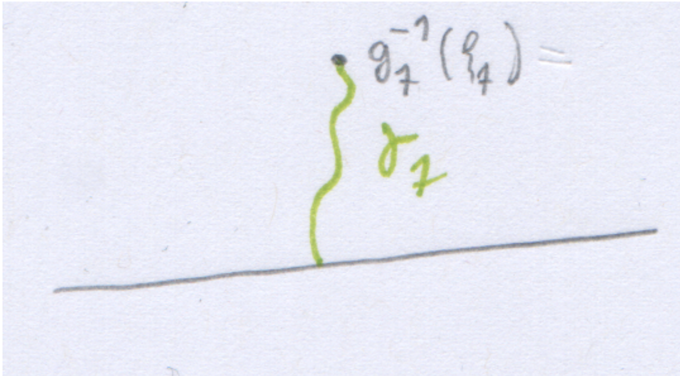

Figure 13: The image of ξ under g^{-1} is the top of the path.

projected onto zero, so we can make use of the conclusions from chapter 3.1 more easily. Define

$$h_t(z) = g_t(z) - \xi_t$$

.

Now what effect do the known properties (Markov, stationarity of increments) have on the driving term ξ_t? The Markov property implies that for $s > t$: $\xi_s - \xi_t$ is independent of $\xi_{t'}$, $t' \leq t$ and the stationarity of increments leads to the ξ_t being distributed exactly like ξ_{s-t}. If we demand two more very technical details, we can come to a solution for xi_t. The first detail is that ξ_t has to have a continuous trajectory (no branching), which comes natural to physicists. The other one is that the probability distribution hidden in ξ_t has to be symmetric under recflectin at the imaginary axis.

With those details out of the way, we can use the following theorem from probatility theory. A one dimensional Markov process with continuous trajectory, stationary increments and reflexion symmetry is proportional to a 1d Brownian motion (see [4]). So there is a real number κ such that $\xi_t = \sqrt{\kappa} B_t$, where B_t is a normalized Brownian motion with covariance $\mathbf{E}[B_s B_t] = min(s,t)$.

Now... what is a Brownian motion? As before I will not prove this theorem, however I can give you a good idea about the properties of Brownian motions.

We start by defining a probability measure on the space of all parametrizations for paths with continuous trajectory $\Omega = \mathcal{C}_0([0,\infty[,\mathbb{R})$. As a basic object for this probability distribution we take heat kernel, which is a centered Gaussian:

$$K(x,t) = \frac{1}{(2\pi t)^{\frac{1}{2}}} exp \frac{-x^2}{2t}$$

It is alreadycorrectly normalized so that $\mu(\Omega) = \int_{\mathbb{R}} dx K(x,t) = 1$, which we need for a stochastic description.

Now as an example we imagine a two dimensional Brownian motion and the corresponding two dimensional heat kernel, which of course is just a two dimensional centered Gaussian. If we now want to know the probability for a particle to take a specific path

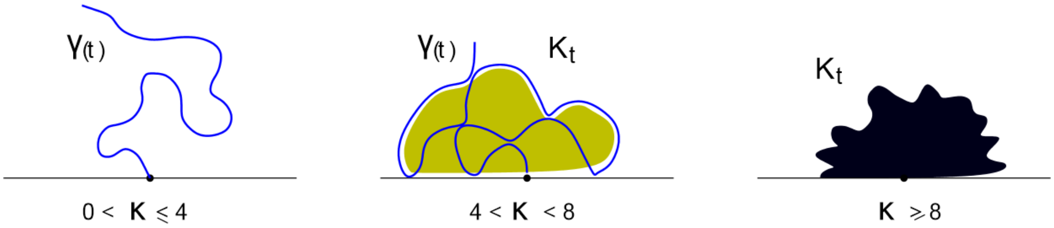

Figure 14: Illustration of the different behavior of SLE for different κ.

in two dimensions, we would have to integrate this centered Gaussion probability distribution over an epsilon neighborhood around that path. Integrating just along the path itself will give us probability zero, since a line has measure zero in two dimensions. So The Markov property and the stationarity of increments mean for the Brownian motion that for all t the probability distribution for the particle is independent of whatever happened before and it stays the same for all times (a centered Gaussian).

Now why is the Brownian motion for our problem just a one dimensional one, the path is also in two dimensions. The answer is to remember that with the driving term xi_t being zero, the path just moves in a straight line. So we have just one degree of freedom left, namely how big is the curvature. So in our case the one dimensional Brownian motion is the probability distribution for the curvature of the path.

With that we can finally conclude the chordal Schramm-Loewner evolution of parameter κ:

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}$$

In Figure 14 three simulations for different κ are shown. The behavior is completely different for changing values of κ. For $0 < \kappa \leq 4$ the curve intersects with itself only a limited number of times and then goes to ∞. For $4 < \kappa < 8$ the curve intersects itself and the real axis an infinite number of times, and in the case of $8 \leq \kappa$ the curve actually fills out the volume above 0. Intuitively this shows that the more power we give the driving Brownian motion by making κ bigger, the more the path curves.

4 From SLE to CFT

4.1 SLE and the Witt algebra

To see the first connection to conformal field theory we want to show how the solutions h_t we found in the last chapter relate to the Witt algebra.

$$l_n = z^{n+1} \partial_z \qquad [l_n, l_m] = (n-m) l_{m+n}$$

For this we view the h_ts as group elements of some group N whichs properties we want to study. This means studying how these group elements act on holomorphic functions $f : V \to W$ (we act on some subspace of the Riemann surface, so we can choose local coordinates). As always, we are more interested in members of the tangent space dh_t with initial condition $h_0(z) = z$ rather than members of the actual group. In this calculation we will have to pay special attention to the stochastic nature of the processh_t,remember ξ_t is a Brownian motion with covariance $E[\xi_t\xi_s] = \kappa min(t, s)$. We can express the members of the tangent space $dh_t(z)$ as a differential depending on dt and $d\xi_t$.

$$dh_t(z) = dt\sigma(h_t(z)) + d\xi_t\rho(h_t(z))$$

Now let the h_t act on the holomorphic functions in the following way

$$h_t^f := f \circ h_t \circ f^{-1}$$

For the differential this means:

$$dh_t^f = dt(\sigma^f \circ h_t^f) + d\xi_t(\rho^f \circ h_t^f)$$

To calculate this we need the stochastik version of the chain rule, called Ito's formula.

$$\rho^f \circ f = f'\rho \qquad \sigma^f \circ f = f'\sigma + \frac{\kappa}{2} f''\rho^2$$

Then we have:

$$dh_t^f = dt((f'\sigma + \frac{\kappa}{2} f''\rho^2) \circ f^{-1} \circ h_t^f) + d\xi_t(f'\rho \circ f^{-1} \circ h_t^f)$$

O is the space of holomorphic functions and the group N acts on it. We define that action as $\mathfrak{g}_h \cdot f = f \circ h$. So then the equation becomes:

$$(\mathfrak{g}_{h_t^f}^{-1} d\mathfrak{g}_{h_t^f})f = dt(f'\sigma + \frac{\kappa}{2} f''\rho^2) + d\xi_t(f'\rho)$$

In the case of the chordal SLE this heavily simplifies to:

$$\mathfrak{g}_{h_t^f}^{-1} d\mathfrak{g}_{h_t^f} = dt(\frac{2}{z}\partial_z + \frac{\kappa}{2}\partial_z^2) - d\xi_t\partial_z$$

Now we define the operators $l_{-1} = -\partial_z$ and $l_{-2} = -\frac{1}{z}\partial_z$ to get to the conclusion.

$$\mathfrak{g}_{h_t^f}^{-1} d\mathfrak{g}_{h_t^f} = dt(-2l_{-2} + \frac{\kappa}{2} l_{-1}^2) + d\xi_t l_{-1}$$

With these first two operators we could reconstruct the rest of the Witt algebra using commutation.

$$[l_n, l_m] = (n - m)l_{m+n}$$

More importantly the equation let's us understand the role of l_{-1} and l_{-2}, for this we integrate out the last term $\mathfrak{g}_{g_t^f} = \mathfrak{g}_{h_t^f} e^{-\xi_t l_{-1}}$ and get

$$\mathfrak{g}_{g_t^f}^{-1} d\mathfrak{g}_{g_t^f} = -2dt(e^{\xi_t l_{-1}} l_{-2} e^{-\xi_t l_{-1}})$$

With this we can see that the vector field l_{-1} drives the Brownian motion whereas l_{-2} specifies the drift.

4.2 The central charge

One can use both SLE or CFT to understand conformally invariant pictures. CFT expresses conformal symmetry through unitary transformations which act on a Hilbert space. For the SLE it should not make a difference if it is displayed in terms of correlation functions on a Hilbert space. So as our final step of connecting SLE with CFT we want to find the central charge of the corresponding Virasoro algebra (compare to the last chapter of [5]). Our ansatz will be that the expectation values which are time invariant in SLE should be time invariant in CFT as well. As a starting point we take the conclusion of the last chapter (this time with capital L, because we interpret the L as members of the Virasoro algebra, so we imagine that a non trivial central expansion can be found)

$$\mathfrak{g}_{h_t^f}^{-1} d\mathfrak{g}_{h_t^f} = dt(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2) + d\xi_t L_{-1}$$

and then take the expectation value on both sides of the equation (the second term dissapears because the Brownian motion is a centered Gaussian):

$$\mathbf{E}\left(\frac{d\mathfrak{g}_{h_t^f}}{dt}\right) = \mathbf{E}(-2L_{-2}\mathfrak{g}_{h_t^f} + \frac{\kappa}{2}L_{-1}^2\mathfrak{g}_{h_t^f})$$

Define $\mathcal{H}^T = -2L_{-2} + \frac{\kappa}{2}L_{-1}^2$, so the time conserved observables are eigenvectors to the eigenvalue zero of $\mathcal{H}^T$:

$$\mathcal{H}^T \cdot \psi = 0$$

Following our ansatz we want these ψs to be annihilated by the L_n $(n > 0)$ too:

$$L_n \cdot \psi = 0$$

To calculate the condition which is needed for this to be the case, we need the commutator:

$$[L_n, \mathcal{H}^T] = (-2(n+2) + \frac{\kappa}{2}n(n+1))L_{n-2} + \kappa(n+1)L_{-1}L_{n-1} - c\delta_{n,2}$$

It follows that for all $n > 3$ the L_n annihilate ψ;

$$L_n\mathcal{H} \cdot \psi = 0$$

However for demanding $L_1\mathcal{H} \cdot \psi = 0$ and $L_2\mathcal{H} \cdot \psi = 0$, two more identities are required

$$2\kappa h = 6 - \kappa \qquad c = h(3\kappa - 8),$$

where h is the conformal dimension $L_0 \cdot \psi = h \cdot \psi$. By eliminating the conformal dimension h, we finally get an expression for the central charge c depending on κ:

$$c_\kappa = \frac{1}{2}(3\kappa - 8)(\frac{6}{\kappa} - 1)$$

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