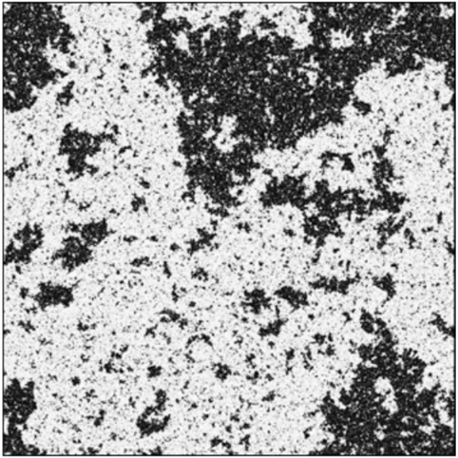
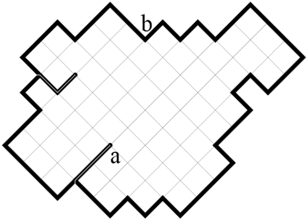
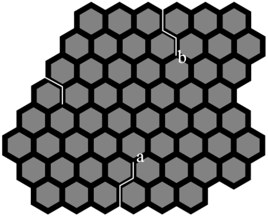
SLE

08/04/2013

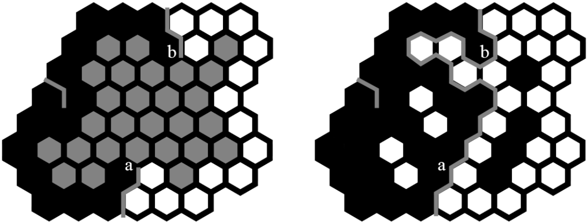


Outline

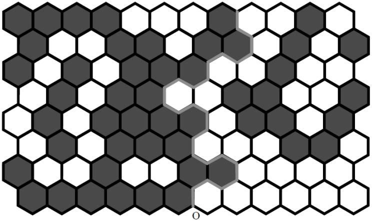




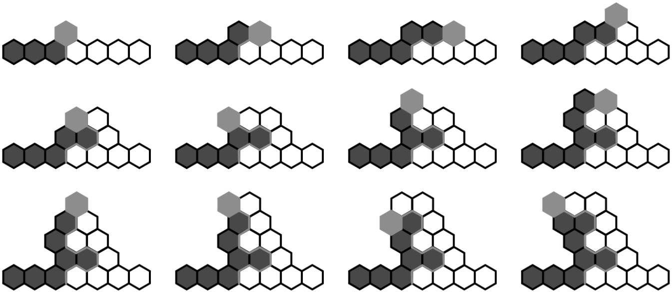
- a hexagonal lattice domain $\mathbb{D}$ is a domain in the usual sense, which can be decomposed as a union of
  - open hexagons with side length 1 (faces)
  - open segments of length 1 (edges)
  - points (vertices)
- an admissible boundary condition is a couple of distinct points (a, b), $a, b \notin \mathbb{D}$, such that a and b can be connected by a path $\gamma \in \mathbb{D}$

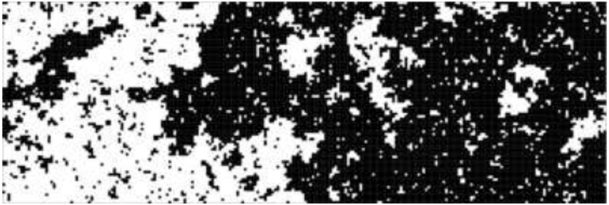


- a path $\gamma \in \mathbb{D}$ from a to b is a sequence $s_1, ..., s_{2n+1}$, where
  - $s_1 = a$ and $s_{2n+1} = b$
  - the s_{2m+1} are distinct vertices
  - the s_{2m} are distinct edges with boundary $\{s_{2m-1}, s_{2m+1}\}$
- $(\mathbb{D}, a, b)$ hexagonal lattice domain with admissible boundary condition



- for a given configuration of right(white) and left(black) hexagons the path from a to b is unique
- example: color the of inner hexagons with a fair coin $p = 1/2$ (independent random variables)
- induced probability distribution for the paths from a to b is called percolation probability distribution
- a path touching l distinct faces has probability 2^{-l}

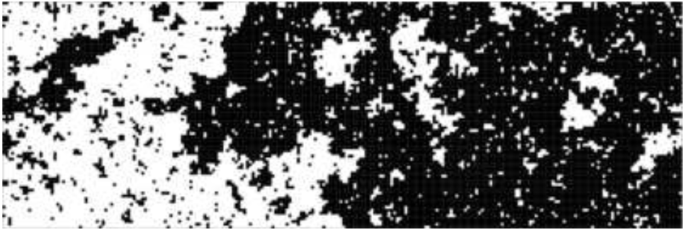




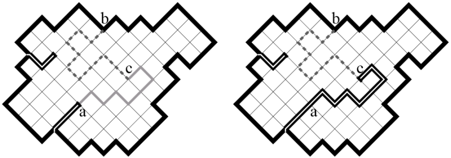
- given $(\mathbb{D}, a, b)$, where the colors of the faces are spin variables σ_i

$$H(\sigma) = -\sum_{<i,j>} J\sigma_i\sigma_j \qquad P_\beta(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z_\beta}$$

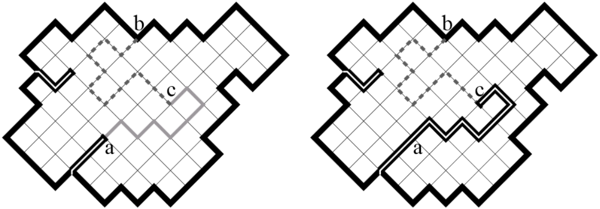
- get exactly one interface (path from a to b) and a number of loops



- adjust J, β until phase transition occurs
  $\longrightarrow$ get a critical system with long range correlations (scale invariance)
- the Ising interaction is local (nearest neighbor)



- given $(\mathbb{D}, a, b)$ and $1 \leq m < n$: obtain hexagonal lattice domain $\mathbb{D}'$ by removing s_l, $1 < l \leq s_{2m+1}$ then (s_{2m+1}, b) is an admissible boundary condition for $\mathbb{D}' \longrightarrow (\mathbb{D}', s_{2m+1}, b)$
- $\gamma_{[a,b]}$ a path from a to b, compare the following probabilities:
  - the probability of $\gamma_{[a,b]}$ in $(\mathbb{D}, a, b)$ given $\gamma_{[a,c]}$ is fixed already
  - the probability of $\gamma_{[c,b]}$ in $(\mathbb{D}', c, b)$



- the Domain Markov property is the statement that these two probabilities are equal

$$P_{(\mathbb{D},a,b)}(.|\gamma_{[a,c]}) = P_{(\mathbb{D}\setminus\gamma_{[a,c]},c,b)}(.)$$

- the DMP is a way to express locality

- continuous limit means
  - lattice spacing goes to 0
  - infinitely many faces in domain $\mathbb{D}$ (every point is a face)
  - paths still do not contribute ($\mu(\gamma) = 0$)
  - a contributing term is an area $\rho(x, y)dxdy$
- domain $\mathbb{D}$ with points a,b on its boundray and $\gamma_{[a,b]}$ connecting them

$$\gamma(0) = a \qquad \gamma(\infty) = b \qquad \gamma_{]a,b[} = \gamma(]0, \infty[) \subset \mathbb{D}$$

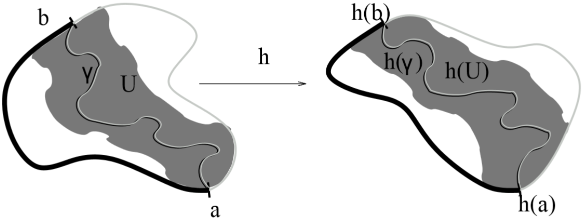
- two domains $\mathbb{D}$ and $\mathbb{D}'$ are always coformally invariant i.e. $\exists$ invertible holomorphic map between them

- want the domain Markov property to hold in the continuous case:

$$P_{(\mathbb{D},a,b)}(.|\gamma_{[a,c]}) = P_{(\mathbb{D}\backslash\gamma_{[a,c[},c,b)}(.)$$

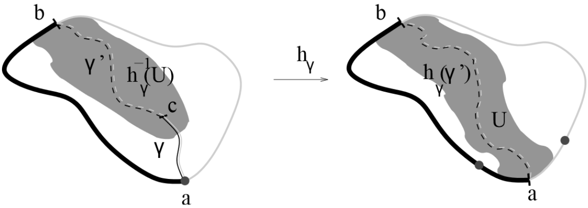
- Conformal transport: conformal map $h : \mathbb{D} \to \mathbb{D}'$ transports the probability measure of the curve γ $(U \subset \mathbb{D})$

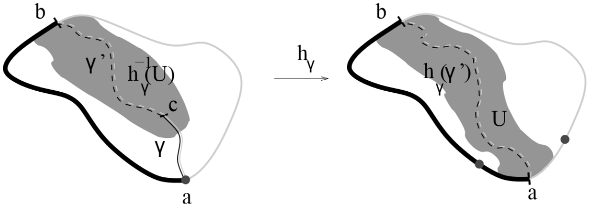
$$P_{(\mathbb{D},a,b)}(\gamma_{[a,b]} \subset U) = P_{(h(\mathbb{D}),h(a),h(b))}(\gamma_{[h(a),h(b)]} \subset h(U))$$



- $(\mathbb{D}, a, b)$, $c \in \mathbb{D}$; $\gamma_{[a,c]}$ a fixed curve
- because of the domain Markov property we know we can cut out $\gamma_{[a,c]}$
- now map $h_{\gamma_{[a,c]}} : \mathbb{D} \backslash \gamma_{[a,c[} \rightarrow \mathbb{D}$ so that:

$$h_{\gamma_{[a,c]}}(\mathbb{D} \backslash \gamma_{[a,c[}) = \mathbb{D} \quad h_{\gamma_{[a,c]}}(c) = a \quad h_{\gamma_{[a,c]}}(b) = b$$



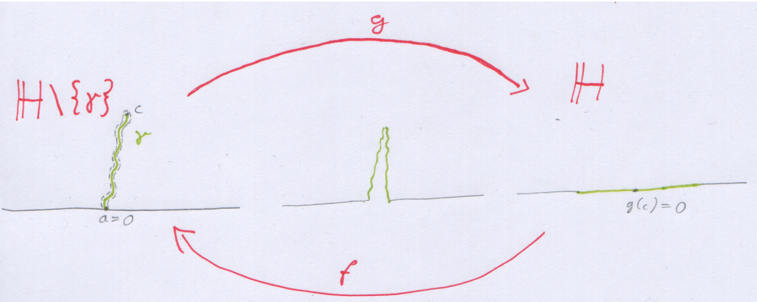


- the resulting equality for the probability is

$$P_{(\mathbb{D},a,b)}(\gamma_{[c,b]} \subset U | \gamma_{[a,c]}) = P_{(\mathbb{D},a,b)}(\gamma_{[a,b]} \subset h_{\gamma_{[a,c]}}(U))$$

- the probability distribution for $\gamma_{[c,b]}$ is
  - independent of $\gamma_{[a,c]}$ (Markov)
  - the same distribution as for $\gamma_{[a,b]}$ (stationarity of increments)

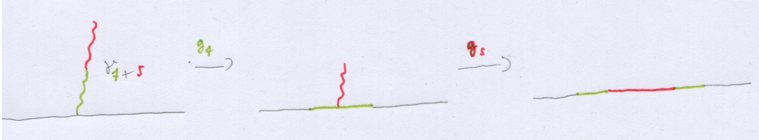
- from now on look at the upper half plane $\mathbb{D} = \mathbb{H}$
- $\gamma(0) = a = 0$; $\mathbb{R} \cup \infty$ is the boundary
- the set $\mathbb{H} \backslash \gamma$ is still a domain (still contractible)
- can map $\mathbb{H} \backslash \gamma$ back onto $\mathbb{H}$



- introduce “time” parameter t for paths γ_t
  $\longrightarrow$ parameter associated to growth of path
- can now chain conformal maps in the following way:

$$g_t : \mathbb{H} \backslash \gamma_t \rightarrow \mathbb{H} \qquad g_s : \mathbb{H} \backslash \gamma_s \rightarrow \mathbb{H}$$

$$g_{t+s} : \mathbb{H} \backslash \gamma_{t+s} \rightarrow \mathbb{H}$$



- want to understand local growth

$$g_t : \mathbb{H}\backslash\gamma_t \to \mathbb{H} \qquad g_{t+\epsilon} : \mathbb{H}\backslash\gamma_{t+\epsilon} \to \mathbb{H}$$

- get a derivative describing local growth

$$\frac{dg_t}{dt} = lim_{\epsilon\to 0}\frac{g_{t+\epsilon} - g_t}{\epsilon}$$

- for a path this is (depends on parameterization)

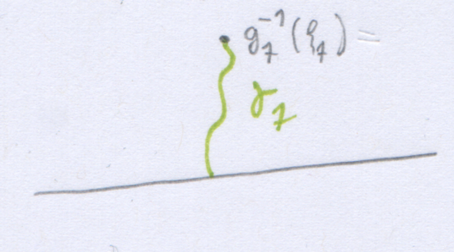
$$\frac{dg_t}{dt}(z) = \frac{2}{g_t(z) - \xi_t}$$

  Loewner chain for simple paths

- the image of ξ_t by g_t^{-1} is the tip of the path γ_t at time t

$$\gamma_t = lim_{\epsilon \to 0} g_t^{-1}(\xi_t + i\epsilon)$$

  so ξ_t provides a parametrization for γ



- the solution $g_t(z)$ to the equation

$$\frac{dg_t}{dt}(z) = \frac{2}{g_t(z) - \xi_t}$$

  for given ξ_t and initial condition $g_0(z) = z$ is called Loewner evolution
- numerically solveable at least for t small enough

- driving term ξ_t is now a random variable and

$$h_t(z) = g_t(z) - \xi_t$$

- what effect do the known properties (Markov, stationarity of increments) have on the random variable ξ_t?
  - for $s > t$: $\xi_s - \xi_t$ is independent of $\xi_{t'}$, $t' \leq t$ (Markov)
  - and distributed like a ξ_{s-t} (stationarity of increments)
- need to demand 2 more things to come to a conclusion
  - ξ_t has a continuous trajectory (no branching)
  - distribution has to be symmetric under reflection at imaginary axis $g_t(z) = -\overline{g_t(-\bar{z})}$

- Theorem: a 1d Markov process with continuous trajectory, stationary increments and reflexion symmetry is proportional to a 1d Brownian motion
- so there is a real number κ such that $\xi_t = \sqrt{\kappa} B_t$, where B_t is a normalized Brownian motion with covariance $\mathbf{E}[B_s B_t] = min(s,t)$

- want to define a probability measure on the space $\Omega = \mathcal{C}_0([0, \infty[, \mathbb{R})$
- take heat kernel (centered Gaussian) as a basic object

$$K(x,t) = \frac{1}{(2\pi t)^{\frac{1}{2}}} exp \frac{-x^2}{2t}$$

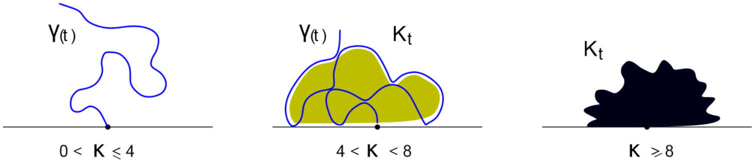
correctly normalized so that $\mu(\Omega) = \int_{\mathbb{R}} dx K(x,t) = 1$

- if we want to know what the probability for the particle to choose one specific path γ is, we can integrate $K(x,t)$ over an ϵ neighborhood of γ

- let the Brownian motion drive a 1d point function $\omega : [0, \infty[\to \mathbb{R}$
- then for $0 < t_1 < ... < t_n$ the vector $(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})$ is centered Gaussian with indepentent components
- the Brownian motion has all the required properties
  - probability distribution for the particle for $t > t_0$ independent of what happened before t_0 (Markov)
  - for every t the current probability distribution for the particle is again a centered Gaussian (stationarity of increments)

- Conclude the chordal Schramm-Loewner evolution of parameter κ

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}$$



- view the $h_t \in N$ as group elements
  want to calculate how these group elements act on holomorphic functions $f : V \to W$ (act on some subspace of the Riemann surface $\to$ can choose local coordinates)
- then the dh_t are elements of the tangent space with initial condition $h_0(z) = z$
- h_t is a stochastik process, ξ_t is a Brownian motion with covariance $E[\xi_t \xi_s] = \kappa min(t, s)$

$$dh_t(z) = dt\sigma(h_t(z)) + d\xi_t \rho(h_t(z))$$

- now let the h_t act on the holomorphic functions:
$h_t^f = f \circ h_t \circ f^{-1}$

$$dh_t^f = dt(\sigma^f \circ h_t^f) + d\xi_t(\rho^f \circ h_t^f)$$

- to calculate this we need Ito's formula (stochastik version of the chain rule)

$$\rho^f \circ f = f'\rho \qquad \sigma^f \circ f = f'\sigma + \frac{\kappa}{2} f''\rho^2$$

- then we have

$$dh_t^f = dt((f'\sigma + \frac{\kappa}{2} f''\rho^2) \circ f^{-1} \circ h_t^f) + d\xi_t(f'\rho \circ f^{-1} \circ h_t^f)$$

- space O of hol. functions, group N acting on it:
$\mathfrak{g}_h \cdot F = F \circ h$

$$(\mathfrak{g}_{h_t^f}^{-1} d\mathfrak{g}_{h_t^f})f = dt(f'\sigma + \frac{\kappa}{2} f''\rho^2) + d\xi_t(f'\rho)$$

- in case of chordal SLE:

$$\mathfrak{g}_{h_t^f}^{-1} d\mathfrak{g}_{h_t^f} = dt(\frac{2}{z}\partial_z + \frac{\kappa}{2}\partial_z^2) - d\xi_t \partial_z$$

- define the operators $l_{-1} = -\partial_z$ and $l_{-2} = -\frac{1}{z}\partial_z$

$$\mathfrak{g}_{h_t^f}^{-1} d\mathfrak{g}_{h_t^f} = dt(-2l_{-2} + \frac{\kappa}{2} l_{-1}^2) + d\xi_t l_{-1}$$

- can reconstruct the rest of the Witt algebra with commutation

$$[l_n, l_m] = (n-m) l_{m+n}$$

- want to understand the role of l_{-1} and l_{-2}, again take

$$\mathfrak{g}_{h_t^f}^{-1} d\mathfrak{g}_{h_t^f} = dt(-2l_{-2} + \frac{\kappa}{2} l_{-1}^2) + d\xi_t l_{-1}$$

- can integrate out the last term $\mathfrak{g}_{g_t^f} = \mathfrak{g}_{h_t^f} e^{-\xi_t l_{-1}}$ and get

$$\mathfrak{g}_{g_t^f}^{-1} d\mathfrak{g}_{g_t^f} = -2dt(e^{\xi_t l_{-1}} l_{-2} e^{-\xi_t l_{-1}})$$

- so the vector field l_{-1} drives the Brownian motion whereas l_{-2} specifies the drift

- can use SLE or CFT to understand conformally invariant pictures
- CFT expresses conformal symmetry through unitary transformations which act on a Hilbert space
- for SLE it should not make a difference if displayed as direct functions or correlation functions on a Hilbert space
- we are now looking for a central charge

- ansatz: expectation values which are time invariant in SLE should be time invariant in CFT as well
- again, take the equation

$$\mathfrak{g}_{h_t^f}^{-1} d\mathfrak{g}_{h_t^f} = dt(-2L_{-2} + \frac{\kappa}{2} L_{-1}^2) + d\xi_t L_{-1}$$

- take the expectation value (second term dissapears because Brownian motion is a centered Gaussian)

$$\mathbf{E}\left(\frac{d\mathfrak{g}_{h_t^f}}{dt}\right) = \mathbf{E}(-2L_{-2}\mathfrak{g}_{h_t^f} + \frac{\kappa}{2} L_{-1}^2 \mathfrak{g}_{h_t^f})$$

define $\mathcal{H}^T = -2L_{-2} + \frac{\kappa}{2} L_{-1}^2$

- look for zero modes of $\mathcal{H}^T$, eigenvectors so that $\mathcal{H}^T \cdot \psi = 0$ $\rightarrow$ zero mode is an observable conserved in mean.
- want the zero modes to be annihilated by the L_n ($n > 0$), i.e. among the highest weight vectors $L_n \cdot \psi = 0$ with conformal dimension d, $L_0 \cdot \psi = d \cdot \psi$
- under which condition is $\mathcal{H}^T \cdot \psi$ again a highest weight vector

$$[L_n, \mathcal{H}^T] = (-2(n+2) + \frac{\kappa}{2} n(n+1)) L_{n-2} + \kappa(n+1) L_{-1} L_{n-1} - c\delta_{n,2}$$

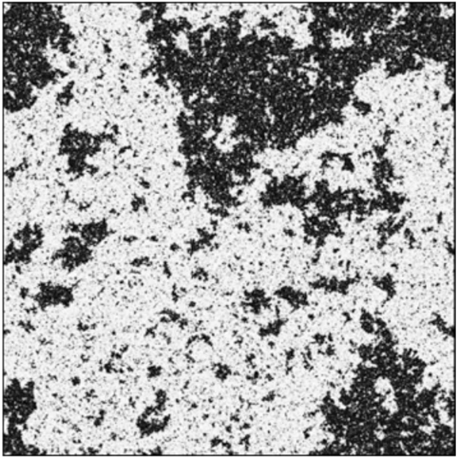
- for all $n > 3$: $L_n \mathcal{H} \cdot \psi = 0$

- but for demanding $L_1\mathcal{H}\cdot\psi = 0$ and $L_2\mathcal{H}\cdot\psi = 0$ it is required that

$$2\kappa h = 6 - \kappa \qquad c = h(3\kappa - 8)$$

- need to adjust the central charge to

$$c_\kappa = \frac{1}{2}(3\kappa - 8)(\frac{6}{\kappa} - 1)$$



Questions?

References:

- Michel Bauer, Dennis Bernhard, *2D growth processes: SLE and Loewner chains* (2008)
- Michel Bauer, Dennis Bernhard, SLE_κ *growth processes and conformal field theory* (2002)