The Heterotic String

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Outline of this presentation

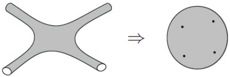
1. Introduction
2. Modular Invariance
3. Construction of the Heterotic String
4. Lie theory and lattices
5. Spectrum

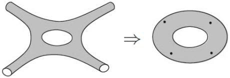
Introduction

- Use many topics of previous talks to construct the heterotic string
  - Lie theory, modular invariance, superstrings, compactification...
- Heterotic string has been believed to be a starting point for reproducing the standard model
  - No tachyon, graviton, gauge symmetry ($\Rightarrow$ interactions)

Part I: Modular Invariance

String perturbation expansion





$$A_n = \sum_{g=0}^{\infty} A_n^{(g)}$$

$$= \sum_{g=0}^{\infty} C_{\Sigma_g} \int \mathcal{D}h \mathcal{D}X^{\mu} \int d^2 z_1 \dots d^2 z_n V_1(z_1, \bar{z}_1) \dots V_n(z_n, \bar{z}_n) e^{-S[X,h]}$$

String perturbation expansion

$$A_n = \sum_{g=0}^{\infty} A_n^{(g)}$$
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From now on: concentrate on the one-loop vacuum amplitude $A_0^{(1)} \sim \int_{\text{Torus}} \mathcal{D}h\mathcal{D}X^{\mu} e^{-S[X,h]}$

Redundancy

Recall: Polyakov action is invariant under Weyl rescalings and diffeomorphisms of the world-sheet

$$\text{Diffeo.}\begin{cases} \delta X^\mu = \xi^\alpha \partial_\alpha X^\mu, \\ \delta h_{\alpha\beta} = -(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha) \end{cases} \qquad \xi: \text{ vector}$$

$$\text{Weyl}\begin{cases} \delta X^\mu = 0 \\ \delta h_{\alpha\beta} = 2\Lambda h_{\alpha\beta} \end{cases}$$

$\Rightarrow$ Because of overcounting, path integral is highly divergent!

Redundancy and Modular Invariance

Try to compensate the overcounting

$$\int \frac{\mathcal{D}h}{\mathrm{Vol(Diff)Vol(Weyl)}}$$

$\Rightarrow$ Integration should be performed on a moduli space of metrics

$$\mathcal{M}_g = \frac{\{\mathrm{metrics}\}}{\{\mathrm{Weyl}\} \times \{\mathrm{diffeomorphisms}\}}$$

and the one-loop partition function must be modular invariant.

Tangent space decomposition

How to perform this in practice?

- Base space: modular parameters.
- Tangent space: Weyl and Diffeo.

$$\delta h_{\alpha\beta} = \underbrace{\delta\Lambda h_{\alpha\beta}}_{\text{Weyl}} + \underbrace{\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha}_{\text{Diffeo.}} + \underbrace{\sum_i \delta\tau_i \frac{\partial}{\partial \tau_i} h_{\alpha\beta}}_{\text{Moduli parameters}}$$

Define operator P (later purpose):

$$(P\xi)_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - (\nabla_\gamma \xi^\gamma) h_{\alpha\beta}$$

Restrict integration to the slice above. Torus: next slide

Moduli space of the torus

Restriction to the slice of modular parameters for one-loop vacuum amplitude $A_0^{(1)}$. World-sheet is a torus. Recall: one modular parameter τ

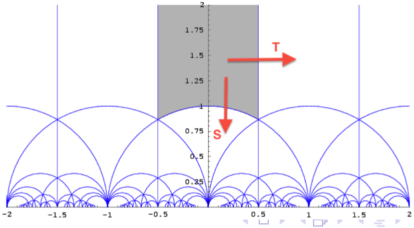
$$\mathcal{M}_1 = \mathbb{H}/\{\text{action of PSL}_2(\mathbb{Z})\}$$

How to obtain that?

- Teichmueller space (conformally inequivalent tori)
- Generators of the modular group (global diffeomorphisms)

$$T : \tau \to \tau + 1$$

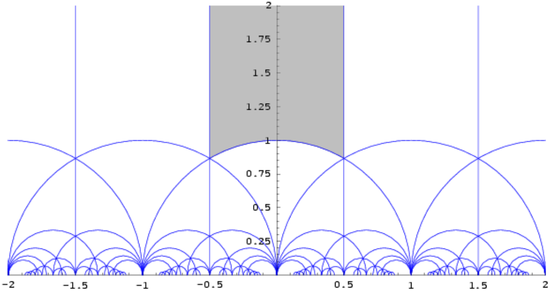
$$S : \tau \to -\frac{1}{\tau}$$



The Fundamental Region

Result: inequivalent metrics reside on the fundamental region $\mathcal{F}$

$$\mathcal{F} = \{z \in \mathbb{H} | |\mathrm{Re}(z)|^2 \leq \frac{1}{2},\ |z| \geq 1\}$$



Orthogonal decomposition

Now, how to divide the measure in a suitable way?

- Need a notion of orthogonality :

$$(\delta h^{(1)}, \delta h^{(2)}) = \int \sqrt{h} h^{\alpha\gamma} h^{\beta\delta} \delta h^{(1)}_{\alpha\beta} \delta h^{(2)}_{\gamma\delta}$$

- Decompose metric othogonally into Weyl+Diffeo+Moduli. Yelds Jacobian $\mathcal{J}$

$$\mathcal{D}h = \mathcal{J}\mathcal{D}\{\text{Weyl}\}\mathcal{D}\{\text{Diffeo}\}d\tau$$

- Would like to cancel out Weyl+Diffeo by further restricting integration on the fundamental region

Conformal Killing Group

Problem: Diffeo. and Weyl overlap. Restricting the integration does not completely eliminate the overcounting.

- Recall operator P:

$$(P\xi)_{\alpha\beta} = \underbrace{\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha}_{\text{Diffeo.}} - \underbrace{(\nabla_\gamma \xi^\gamma) h_{\alpha\beta}}_{\text{Weyl.}}$$

- Zero modes: "Conformal Killing Vectors". Form the "Conformal Killing Group".

$\Rightarrow$ have to divide integration measure by the "volume" of the CKG.

Conformal Killing Group of the Torus

- Conformal Killing Group of the Torus (CKG): $U(1) \times U(1)$.
- Generators: vector fields ∂_z and $\partial_{\bar{z}}$
- Volume:

$$\mathrm{Vol}(\mathrm{CKG}) \sim \mathrm{Im}(\tau)$$

- Nice remark: by the Riemann-Roch theorem, $\dim_{\mathbb{C}}(\mathrm{CKG}) = \{\text{number of moduli parameters}\}$

Finally: the integration measure

From Vol(CKG) and zero-mode Jacobians, one obtains

$$\int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\tau)^3}$$

However, measure by itself not modular invariant. Let $\tau \to \frac{a\tau+b}{c\tau+d}$ be an arbitrary modular transformation. Then:

$$d^2\tau \to |c\tau + d|^{-4} d^2\tau$$
$$\mathrm{Im}(\tau) \to |c\tau + d|^{-2} \mathrm{Im}(\tau)$$

$\Rightarrow$ Only $\frac{d^2\tau}{(\mathrm{Im}\tau)^2}$ is modular invariant.

The integrand

Recall $A_0^{(1)} \sim \int_{\mathrm{Torus}} \mathcal{D}h\mathcal{D}X^\mu e^{-S[X,h]}$. We have decomposed the integration on the metrics. What obout the rest?

- Recall that partition function of CFT's on a Torus corresponds to the generating functional of a QFT with time compactified on a circle of radius $R = \frac{1}{T}$ (temperature).
- Here: similar situation.

Nevertheless, should take some care with zero-modes and contributions
$(i.e.\ \int \frac{d^D}{(2\pi)^D} \langle p| \, e^{-\pi\alpha \mathrm{Im}(\tau) p^2} \, |p\rangle \sim \frac{1}{\sqrt{\mathrm{Im}(\tau)}})$
and non-zero modes of Jacobian

CFT and String one-loop partition function

Let $Z^*(\tau,\bar{\tau})$ denote the usual CFT partition function counting contributions in light-cone gauge but without those of zero-modes. Then:

$$A_0^{(1)} \sim \int_{\mathcal{F}} \underbrace{\frac{d^2\tau}{(\mathrm{Im}\tau)^2}}_{\text{Modular invariant}} \underbrace{\mathrm{Im}(\tau)^{-\frac{D}{2}+1} Z^*(\tau,\bar{\tau})}_{\text{Z}}$$

where D is the number of non-compact dimensions.

For any sting theory, $Z(\tau,\bar{\tau})$ as defined above must be modular invariant.

Part II: Construction of the Heterotic String

Motivation

Recall: would like to reproduce the Standard Model.
Starting point:

- Superstrings have no tachyons, contain bosons and fermions
- Would like another feature: interactions. Try to implement gauge symmetries.

Note/recall: left- and right-moving sectors of the string are independent. E.g.

$$[\bar{\alpha}^{\mu}_{m}, \alpha^{\nu}_{n}] = 0$$

Idea: try to combine bosonic string and superstring.

Basic idea

- Take bosonic right-moving sector and supersymmetric left moving sector
- Recall: anomalies cancel in different space-time dimensions (26 and 10)
- Match dimensions by compactification!
  $\rightarrow$ Compactify 16 bosonic dimensions on a torus

Result:

- Result: String theory in 10D with gauge symmetry

Coordinates of the Heterotic String

1. Left-moving coordinates
   - 10 uncompactified bosonic fields $X_L^\mu(\tau+\sigma)$, $(\mu = 0, ..., 9)$
   - 16 internal bosons $X_L^I(\tau+\sigma)$ $(I = 1, ..., 16)$ living on a torus
2. Right-moving coordinates
   - 10 uncompactified bosons $X_R^\mu(\tau-\sigma)$, $(\mu = 0, ..., 9)$ with their
   - fermionic superpartners $\Psi_R^\mu(\tau-\sigma)$

How do the compactified coordinates look like? Consider compactified space (next slide).

Internal coordinates: discretized momenta

Recall one coordinate: single valuedness of the wave function $exp(ixp) \Rightarrow$ discretized momenta

$$X^{25} \sim X^{25} + 2\pi RL, \;\; L \in \mathbb{R}$$
$$p^{25} = \frac{M}{R}, \; M \in \mathbb{Z}$$

Here: 16 coordinates

$$X^I \sim X^I + 2\pi \sum_{i=1}^{D} n^i e_i^I = X^I + 2\pi L^I, \;\; n_i \in \mathbb{Z}$$

where the $\{e_i\}_{i=1...D}$ are basis vectors of a lattice Λ.
Momenta of additional bosons must be vectors of its 16-dimensional dual lattice $\Gamma_{16} = \Lambda^*$.

Basic definitions of lattices

Definition: Lattice

A *n-dimensional lattice* Γ_n is a set of points in $\mathbb{R}^n$ which can be written as integer combination of a set of basis vectors $\Gamma_n = \{x = \sum x^i e_i | x^i \in \mathbb{Z}\}$

Definition: Dual lattice

The *dual lattice* Γ_n^* is the lattice defined as $\Gamma_n^* = \{y | (y, x) \in \mathbb{Z},\ x \in \Gamma_n\}$

Definition: Even lattice

A lattice is called *even* if for any two vectors x, $y \in \Gamma$, $(x, y) \in 2\mathbb{Z}$.

one-loop partition function

Recall Virasoro characters for bosons and fermions:

$$\chi_{8-\text{fermions}}(\tau) = \frac{1}{2}\frac{1}{|\eta(\tau)|^4}\left(\theta(\tau)_3^4 - \theta(\tau)_4^4 - \theta(\tau)_2^4\right)$$

$$\chi_{n-\text{bosons}}(\tau) = \left(\frac{1}{|\eta(\tau)|}\right)^n$$

In addition, compactified bosons ($q = e^{2\pi i\tau}$):

$$\chi_{16-\text{comp.bosons}}(\tau) = \frac{1}{|\eta(\tau)|}^{16} \sum_{\mathbf{p}_L \in \Gamma_{16}} q^{\frac{1}{2}\mathbf{p}_L^2}$$

one-loop partition function

$$Z^*_{\text{het}}(\tau,\bar\tau) = \chi_{8-\text{fermions}}(\tau)\chi_{8-\text{bosons}}(\tau)\chi_{8-\text{bosons}}(\bar\tau)\chi_{16-\text{comp.bosons}}(\bar\tau)$$

$$= \frac{1}{|\eta(\tau)|^4}\left(\theta(\tau)_3^4 - \theta(\tau)_4^4 - \theta(\tau)_2^4\right)\left(\frac{1}{|\eta(\tau)|}\right)^8\left(\frac{1}{|\eta(\bar\tau)|}\right)^8$$

$$\times \frac{1}{|\eta(\bar\tau)|^{16}}\sum_{\mathbf{p}_L\in\Gamma_{16}} \bar q^{\frac{1}{2}\mathbf{p}_L^2}$$

$$= \left(\frac{1}{[\eta(\bar\tau)]^{24}}\sum_{\mathbf{p}_L\in\Gamma_{16}} \bar q^{\frac{1}{2}\mathbf{p}_L^2}\right)\left(\frac{1}{[\eta(\tau)]^{12}}\left(\theta_3^4(\tau) - \theta_4^4(\tau) - \theta_2^4(\tau)\right)\right)$$

one-loop partition function

By the previous discussion, the full partition function (with zero modes!)

$$Z_{het}(\tau,\bar{\tau}) = \frac{1}{(\mathrm{Im}\tau)^4}\left(\frac{1}{[\eta(\bar{\tau})]^{24}}\sum_{\mathbf{p}_L\in\Gamma_{16}}\bar{q}^{\frac{1}{2}\mathbf{p}_L^2}\right)\times$$
$$\times\left(\frac{1}{[\eta(\tau)]^{12}}\left(\theta_3^4(\tau)-\theta_4^4(\tau)-\theta_2^4(\tau)\right)\right)$$

has to be modular invariant.

Modular Invariance

Recall: generators of the modular group

$$T : \tau \to \tau + 1$$
$$S : \tau \to -\frac{1}{\tau}$$

We know

	$T : \tau \to \tau + 1$	$S : \tau \to -\frac{1}{\tau}$
$\eta(\tau)$	$e^{\frac{i\pi}{12}}\eta(\tau)$	$\sqrt{-i\tau}\eta(\tau)$
$\theta_2(\tau)$	$e^{\frac{\pi i}{4}}\theta_2(\tau)$	$\sqrt{-i\tau}\theta_4(\tau)$
$\theta_3(\tau)$	$\theta_4(\tau)$	$\sqrt{-i\tau}\theta_3(\tau)$
$\theta_4(\tau)$	$\theta_3(\tau)$	$\sqrt{-i\tau}\theta_2(\tau)$

Modular Invariance

- Only term of Z_{het} whose transformation we don't know is the "soliton sum"

$$P(\tau) \equiv \sum_{\mathbf{p}_L \in \Gamma_{16}} q^{\frac{1}{2}\mathbf{p}_L^2}$$

- Requiring modular invariance of Z_{het} leads to

$$P(\tau + 1) = P(\tau)$$
$$P\left(-\frac{1}{\tau}\right) = \tau^8 P(\tau)$$

- translates into constraints on the allowed lattices!

The lattice Γ_{16}

Claim

Γ_{16} must be an even, self-dual lattice

The lattice Γ_{16}

Theorem

In 16 dimensions, the only even, self-dual lattices are the direct product lattice $\Gamma_{E_8} \times \Gamma_{E_8}$, where Γ_{E_8} is the root lattice of E_8, and $\Gamma_{D_{16}}$, the Lie algebra lattice of $SO(32)$ with the (0) and (S) conjugacy classes (or weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$)

Part III: Lie Theory and Lattices

Root lattice Λ_R

Let g be a Lie-algebra. Recall:

- Cartan subalgebra: set of commuting generators H^I
- Diagonalize remaining generators E^α with respect to its elements

$$[H^I, E^\alpha] = \alpha^I E^\alpha$$

- Vectors α^I are called *roots*

Arbitrary integer linear combinations of roots $\Rightarrow$ root lattice Λ_R.

Weight lattice Λ_w

- Take a particular representation of a Lie Group G. States can be denoted by

$$|\mathbf{m}_I, D\rangle \;\; I \in \{1...D\}$$

D: the dimension of the representation.

- Eigenstates of the Cartan subalgebra generators

$$H^I |c, D\rangle = m^I_I |\mathbf{m}_I, D\rangle$$

- m_I are eingevalues of the H^I: *weight vectors*.

Arbitrary integer linear combinations of weight vectors $\Rightarrow$ weight lattice Λ_w.

The Lie-algebra lattice

Observation:

- $\Lambda_R \subset \Lambda_w$
- $\Lambda_R = \Lambda_w^*$
- $\mathrm{vol}(\Lambda) = \mathrm{vol}(\Lambda^*)^{-1}$

Overall note that

- $\Lambda_w = \Lambda_R \oplus (\Lambda_R + \mathbf{m}_2) \oplus \ldots \oplus \ldots (\Lambda_R + \mathbf{m}_{N_C})$, where $\{\mathbf{m}_i\}_{i=1\ldots N_c}$ are representatives of conjugacy classes

Take only a subset of the conjugacy classes, closed under addition of all lattice vectors ⇒ Lie algebra lattice

E_8 and Γ_{E_8}

What is E_8?

- An exceptional simple simply-laced Lie algebra
- Has dimension 248, rank 8
- Has only one conjugacy class $\Rightarrow$ Γ_{E_8} self-dual

What are its root vectors?

- 112 (8 dimensional) root vectors of D_8

$$(\ldots \pm 1, \ldots, \pm 1, \ldots) \text{ all other entries } 0$$

- following 128 (8 dimensional) vectors

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right) \text{ even number of " } - \text{ " signs}$$

$\mathrm{Spin}(32/\mathbb{Z}_2)$ and $\Gamma_{D_{16}}$

- $\mathrm{Spin}(32)/\mathbb{Z}_2$ is the double cover of $SO(32)$ $\Rightarrow$ they have same dimension
- Recall that Lie algebra of SO(32), denoted D_{16}, has four conjugacy classes: trivial (0), Vector (V), Spinor (S), Conjugate spinor (C) Weight vectors of $\mathrm{Spin}(32)/\mathbb{Z}_2$:
- (0) $\Leftrightarrow$ root lattice of $SO(32)$

$$(k_1 ... k_n)\,, \;\; k_i \in \mathbb{Z},\; \sum_{i=1}^{n} k_i = \mathrm{even}$$

- (S): $\mathbf{m} = \left(\pm\frac{1}{2}, \pm\frac{1}{2}, ... \pm\frac{1}{2},\right)$, with an even number of "-" signs

$\mathrm{Spin}(32/\mathbb{Z}_2)$ and $\Gamma_{D_{16}}$

How to see that $\Gamma_{D_{16}}$, Lie algebra lattice of $SO(32)$, is self-dual?
Consider weight lattice Λ_w and root lattice Λ_R of $SO(32)$

- $\Lambda_w = \Lambda_R + 0_{(0)} \oplus (\Lambda_R + \mathbf{m}_{(S)}) \oplus (\Lambda_R + \mathbf{m}_{(V)}) \oplus (\Lambda_R + \mathbf{m}_{(C)})$
- $\Lambda_{\Gamma_{D_{16}}} = \Lambda_R + 0_{(0)} \oplus (\Lambda_R + \mathbf{m}_{(S)})$
- $\mathrm{vol}(\Lambda_w) = \frac{1}{4}\mathrm{vol}(\Lambda_R),\ \mathrm{vol}(\Lambda_w) = \mathrm{vol}(\Lambda_R)^{-1}$
- $\Rightarrow \mathrm{vol}(\Lambda_w) = \frac{1}{2}$, $\mathrm{vol}(\Lambda_R) = 2$
- $\mathrm{vol}(\Gamma_{D_{16}}) = \frac{1}{2}\mathrm{vol}(\Lambda_R) = 1 \Rightarrow$ unimodular
- Consider vectors of (S): it is integer
- self-dual $\Leftrightarrow$ unimodular and integer

Part IV: Spectrum

Spectrum and the level matching condition

- Spectrum is constructed by taking the tensor product of right- and left-moving excitations
- Right-moving sector: $\mathcal{N} = 1$ supersymmetric in 10 dimensions
- Level matching condition

$$m_L^2 = m_R^2 \leftrightarrow N_L + \frac{1}{2}\mathbf{p}_L^2 - 1 = \begin{cases} N_R & \text{R sector} \\ N_R - \frac{1}{2} & \text{NS sector} \end{cases}$$

The Massless States

1. Components of graviton, antisymmetric tensor and dilaton (NS-sector):
   $\bar{\alpha}^{\mu}_{-1} \left|0\right\rangle \otimes b^{\nu}_{-\frac{1}{2}} \left|0\right\rangle_{NS}$
2. Supersymmetric partners gaugino, dilatino (R-sector):
   $\bar{\alpha}^{\mu}_{-1} \left|0\right\rangle \otimes \left|S^{\alpha}\right\rangle_{R}$
3. The gauge bosons of $E_8 \times E_8$ or $SO(32)$
   - $\bar{\alpha}^{I}_{-1} \left|0\right\rangle \otimes \left|S^{\alpha}\right\rangle_{R}$ Gauge bosons of the Cartan subalgebra
   - $\left|\mathrm{p}_L^2 = 2\right\rangle \otimes \left|S^{\alpha}\right\rangle_{R}$ Root vectors
4. 496 supersymmetric partners, gaugini:
   $\bar{\alpha}^{I}_{-1} \left|0\right\rangle \otimes \left|S^{\alpha}\right\rangle_{R}$ and $\left|\mathrm{p}_L^2 = 2\right\rangle \otimes \left|S^{\alpha}\right\rangle_{R}$

Conclusions

- Singled out two distinct compactifications (on a torus) for the heterotic string from the modular invariance of the one-loop vacuum amplitude
- Many attempts to compactify on other manifolds/orbifolds
- Does any other heterotic String Theory reproduce the Standard Model? Unfortunately, not yet completely

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Thank you!