1 Integrability

As soon as physicists wrote down mathematical equations describing physical systems, they have been looking for situations in which they could be solved exactly. Of course, as you all well know, this is generically impossible. Just think about the weather or some other complicated system like non-perturbative effects in QCD.

Of course there are cases where we do know exact solutions solutions, like free fields or the harmonic oscillator. These models, which we can study in detail teach us a lot about our theory. They can be used to understand mechanisms and give some insight. Of course it is a good idea to first study an idealized situation and understand it before we go and tackle more realistic problems.

Integrable systems are systems for which, in one way or another, we can exactly describe the solutions. In this course we will more or less follow aspects of exactly solvable models through various stages in history. We start with classical mechanics where a lot of important concepts already appear. Then we move on to classical field theory, or partial differential equations. For example, the KdV equation describing waves in shallow water.

After this we continue to the quantum case. First we will do quantum mechanics, where we discuss spin chains, models of magnetism. We finish with quantum field theories. Along the way we hope to illustrate the different concepts by using a lot of examples.

These are some very rough lecture notes regarding the material covered in the first three lectures. For more details we refer the students to the literature listed on the course website.

2 Hamiltonian Mechanics and Integrability

Let us start with classical mechanics. We consider, say, n particles moving in a potential V. At any point in time, the system is fully described by specifying the positions q^i and momenta p_i of the particles. Thus, any configuration corresponds to a point in phase space $\mathcal{M}$ where q, p take values. Letting the system evolve over time then describes a curve in this phase space $(p(t), q(t))$.

The evolution over time is described by the Hamiltonian $H(p, q)$, which is a smooth function on $\mathcal{M}$. As always, we denote the time derivative by a dot. Then the Hamilton equations of motion are

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad\qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{1}$$

Or, written a bit more compactly

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} \qquad\qquad \dot{x} = J\nabla_x H. \tag{2}$$

The matrix J defines a Poisson bracket via $\{F, G\} = \langle \nabla F, J\nabla G\rangle$, which is written in coordinates as

$$\{F, G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q^i}\frac{\partial F}{\partial p_i}. \tag{3}$$

We take the Poisson bracket to be non-degenerate, *i.e.* if $\{F, G\} = 0$ for all G then F is constant. The Poisson bracket satisfies the following properties

bilinearity $\{F, \lambda G + \mu H\} = \lambda\{F, G\} + \mu\{F, H\}$,

skew-symmetry $\{F, G\} = -\{G, F\}$,

Jacobi identity $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$,

Leibniz rule $\{F, GH\} = G\{F, H\} + \{F, G\}H$.

It makes the smooth functions on M into an infinite dimensional Lie algebra. Because of these rules it is completely defined on the basis elements

$$\{x^i, x^j\} = J^{ij}, \qquad \{q^i, q_j\} = \{p_i, p_j\} = 0, \qquad \{p^i, q_j\} = \delta^i_j. \tag{4}$$

Then, the equations of motion simply become

$$\dot{q} = \{H, q\}, \qquad \dot{p} = \{H, p\}, \tag{5}$$

and, in general, by the chain rule of differentiation we have

$$\dot{F} = \{H, F\}, \tag{6}$$

for all functions on $\mathcal{M}$. In particular, the Hamiltonian is time independent and the motion takes place on the submanifold where $H = E$ is constant.

Now suppose we apply a coordinate transformation $x \to y$. We want to find out under which conditions the system remains Hamiltonian. For the equations of motion we find

$$\dot{y}^i = \Lambda^i_k \dot{x}^k = \Lambda^i_k J^{km} \nabla^x_m H = \Lambda^i_k J^{km} \Lambda^n_m \nabla^y_n H, \tag{7}$$

where $\Lambda = \frac{\partial y}{\partial x}$. Or, simply written in matrix form

$$\dot{y} = \Lambda J \Lambda^t \cdot \nabla^y H. \tag{8}$$

Hence the new equations of motion are Hamiltonian if and only if

$$\Lambda J \Lambda^t = J, \tag{9}$$

with the new Hamiltonian $\tilde{H}(y) = H(x(y))$. Transformations of this form are called canonical. In case Λ does not depend on x, then it is part of the Lie group $Sp(2n, \mathbb{R})$, which is called the symplectic group.

Thus the natural way of formulating Hamiltonian mechanics is the language of symplectic geometry. The Poisson bracket is equivalent to the closed two-form

$$\omega = dq \wedge dp, \tag{10}$$

corresponding to the bilinear form $\omega(x, y) = \langle x, J^{-1} y\rangle$. A manifold with a non-degenerate closed two-form is called a symplectic manifold. Thus we consider our phase space to be a symplectic manifold.

In this language we can introduce the vector field X_f corresponding to a function f defined implicitly by the equation

$$df = \omega(X_f, \cdot) = \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i}. \tag{11}$$

Then we can write the Poisson bracket is simply as

$$\{F, G\} := \omega(X_F, X_G). \tag{12}$$

A canonical transformation is a transformation that leaves ω invariant. More explicitly, let $(p, q) \to (p', q')$, then

$$\omega \to \omega' = dq' \wedge dp'. \tag{13}$$

The Hamiltonian H is a smooth function on $\mathcal{M}$ and the Hamiltonian vector field X_H corresponding to the Hamiltonian generates the flow on the manifold.

2.1 Integrals of motion

The key concept in the language of integrability is that of an integral of motion. Consider a set of first order differential equations. The equations of motion in the Hamiltonian formalism are exactly of this form. In particular, we write in coordinates

$$\dot{x} = f(x). \tag{14}$$

Definition A time-independent (first) integral of motion for a Hamiltonian system is a smooth function I defined on an open subset of $\mathcal{M}$ such that $\dot{I} = 0$ on solutions[1].

Equivalently, the integral of motion satsifies $\{H, I\} = 0$. Let $I_1, \ldots, I_k$ be first integrals defined on U_i with $U = \cap U_i$ non-empty. We call them independent if on U

$$\operatorname{rank}(\partial_x I_1, \ldots, \partial_x I_k) = k, \tag{15}$$

in particular their tangent vectors are simply independent. In general integrals of motion can belong to a wide class of functions. For example, in the Lotka-Volterra ABC system

$$\dot{x} = x(Cy + z), \qquad \dot{y} = y(x + Az), \qquad \dot{z} = z(Bx + y) \tag{16}$$

we encounter the following different integrals of motion

$$(A, B, C) = (-\tfrac{1}{2}, -\tfrac{1}{2}, 1), \qquad I = x^2 + y^2 + \tfrac{1}{4}z^2 - 2xy + yz + xz \tag{17}$$

$$(A, B, C) = (1, 1, 1), \qquad I = y^{-1}(x - y)(y - z) \tag{18}$$

$$(A, B, C) = (1, 1, 0), \qquad I = xy^{-1} + \log 1 - yz^{-1} \tag{19}$$

$$(A, B, C) = (1, \sqrt{2}, 1), \qquad I = \frac{z(y - z)^{\sqrt{2}+1}}{xy^{\sqrt{2}}}. \tag{20}$$

We see polynomial, rational, logarithmic and even transcendental functions appearing. We will in what follows consider analytic first integrals, meaning that they can locally be expanded as a convergent power series.

Theorem 2.1 (Poisson) *Let $I_{1,2}$ be two first integrals of motion, then $\{I_1, I_2\}$ is also a (possibly trivial) first integral.*

This possibly creates a new integral. Two integrals are said to be in involution if they Poisson commute, *i.e.* $\{I_1, I_2\} = 0$.

Example Consider a Hamiltonian H of a particle in three dimensions, which has the first two components of the angular momentum $J_{1,2}$ as first integrals. Then by Poisson's theorem J_3 is also an integral of motion. On the other hand, if we redefine $I_1 = H, I_2 = J_3, I_3 = |J|^2$, then we find three first integrals that are in involution.

If our system has an integral of motion, we can obviously restrict ourselves to the level set of that integral of motion and we have effectively removed two degrees of freedom (one p and one q). Thus integrals of motion allow us to reduce the problem to a lower dimensional one. Doing this a sufficient number of times leads to the notion of integrability.

The question how to find first integrals is basically something for which there is no complete algorithm. Usually they are simply found by making an appropriate ansatz. For example, when your Hamiltonian is polynomial, you also expect this form for you conserved quantities.

[1]There is also the notion of second, third etc. integral of motion. These are constant only on certain solutions of the equations of motion.

2.2 Integrability

Finding exact solutions of a dynamical Hamiltonian system is very difficult. Even though in classical mechanics classes you are often only shown exactly solvable examples (harmonic oscillator, planetary motion etc.), an exact solution generically does not exist. Let us specify what we mean by an 'exact' solution. A system is said to be *solved by quadratures* if the solution can be constructed by

- solving a finite number of algebraic equations
- computing a finite number of integrals.

Of course, in practise, obtaining a nice closed form for the solution can then still be very hard.

Example The easiest example is a simple one-dimensional harmonic oscillator with Hamiltonian

$$H = \tfrac{1}{2}(p^2 + \omega^2 q^2). \tag{21}$$

The corresponding equations of motion are

$$\dot{p} = \omega^2 q, \qquad \dot{q} = -p, \tag{22}$$

with the obvious solution

$$q = A \sin \omega t + B \cos \omega t. \tag{23}$$

Without loss of generality let us set $B = 0$ and look at the orbits in phase space. The phase space simply is $\mathbb{R}^2$ and the solution defines the curve

$$(q, p) = A(\sin \omega t, \omega \cos \omega t), \tag{24}$$

which is an ellipse. The energy for this solution is $E = A^2$ is conserved and we see that the energy determines the radius of the ellipse. Thus, the phase space is fibered into ellipses $H = E$.

This is a general, or perhaps defining, feature of integrable systems. There should be conserved quantities that specify the curve in the phase space. This leads to the following definition

Definition Consider a Hamiltonian system with phase space of dimension $2n$, then it is called *Liouville integrable* if it possesses n independent conserved quantities F_i, which are in involution

$$\{F_i, F_j\} = 0. \tag{25}$$

Conventionally, the Hamiltonian is taken to be the first conserved charge $H = F_1$. Independence of the functions means that at any point the gradient vectors are independent.

Of course, any one-dimensional system is integrable by this definition. And indeed, considering for example the harmonic oscillator we can write

$$t = \int \frac{dq}{\sqrt{2E - \omega^2 q^2}}, \tag{26}$$

providing the exact solution by just specifying the energy. There is a theorem that states that any integrable system is solvable by quadratures.

Theorem 2.2 (Liouville) *Any integrable system is solvable by quadratures. Furthermore, if the level set $\mathcal{M}_f = \{x \in \mathcal{M} \,|\, F_i(x) = f_i\}$ is compact and connected, it is diffeomorphic to the n-dimensional torus torus T^n, where the dimension of the phase space is $2n$.*

Proof The idea is to construct a canonical coordinate transformation by using the conserved quantities such that the equations of motion can be trivially solved.

Since we have exactly n conserved quantities we can solve $F_i(p,q) = p$ for p. Consider the canonical one form

$$\alpha = p\, dq, \qquad\qquad \omega = d\alpha. \tag{27}$$

This gives a well-defined one-form on $\mathcal{M}_f$ since we have written $p = p(f,q)$. We then define the so-called generating function S

$$S \equiv \int_{m_0}^{m} \alpha, \tag{28}$$

where we integrate along some path on $\mathcal{M}_f$. In order to show that this is well-defined, we need to show that it is independent of the chosen path. Let us first show how S can be used to find the required canonical transformation.

First, we can define p and a new angle coordinate as partial derivatives of the generating function

$$p = \frac{\partial S}{\partial q}, \qquad\qquad \psi = \frac{\partial S}{\partial f}. \tag{29}$$

Then the coordinate transformation $(p,q) \to (f,\psi)$ is canonical, since

$$0 = d^2 S(f,q) = dp \wedge dq + d\psi \wedge df. \tag{30}$$

Moreover, the equations of motion simply reduce to

$$\dot{F}_i = \{H, F_i\} = 0, \qquad\qquad \dot{\psi}_i = \{H, \psi_i\} = \frac{\partial H}{\partial f_i} \equiv \omega_i = \text{constant}. \tag{31}$$

This means that in these coordinates, the equations of motion are trivially solved since F is constant and ψ is simply linear in time.

Let us now show invariance of S. Suppose that $\mathcal{M}_f$ does not have non-trivial cycles. Then by the Stokes theorem

$$\Delta S = \int_{m_0}^{m_0} \alpha = \int d\alpha = \int \omega = 0, \tag{32}$$

since ω vanishes on $\mathcal{M}_f$. This is because the tangent space to $\mathcal{M}_f$ is generated by the Hamiltonian vector fields X_{F_i}, which are independent. Moreover, since they mutually commute we find $\omega|_{\mathcal{M}_f} = 0$.

In case the manifold has non-trivial cycles, then finds that the change in S after integrating over a cycle is given by

$$\Delta_{cycle} S = \int_{cycle} \alpha, \tag{33}$$

which is a function of F_i only (and hence constant on $\mathcal{M}_f$). In other words, the variables ψ are multi-valued.

Finally, let us show that $\mathcal{M}_f$ has the topology of a torus. Since all the functions F_i are in involution, they define an action of $\mathbb{R}^n$ on $\mathbb{M}_f$ via flows. Since all the vector fields are well-defined and independent, this action is free and transitive. Thus we find a surjective map form $\mathbb{R}^n$ to $\mathcal{M}_f$. Moreover, the stabilizer froms an Abelian subgroup, which in $\mathbb{R}^n$ is a lattice. Hence, the manifold is diffeomorphic to the torus.

Notice that by defining S we have reduced the system to one quadrature and a trivial integration. Explicitly, to get S you have to solve p in terms of F. In all known examples, the conserved quantities are algebraic in p and thus it simply reduces to solving a set of algebraic equations.

Example Let us apply the various notions and steps of the proof to the harmonic oscillator. First we have to find the generating function

$$S(E,q) = \int_0^q \sqrt{E-\omega^2\tilde{q}^2}d\tilde{q} = \frac{1}{2\omega}\left\{q\omega\sqrt{E-\omega^2\tilde{q}^2} + A\arctan\left[\frac{\omega q}{\sqrt{E-\omega^2 q^2}}\right]\right\} \tag{34}$$

Now let us consider the new variables that this generates. We should get the angle and the momentum. We find

$$\frac{\partial S}{\partial q} = \sqrt{E-\omega^2\tilde{q}^2} = p \qquad\qquad \psi = \frac{1}{2\omega}\arctan\left[\frac{\omega q}{\sqrt{E-\omega^2 q^2}}\right] = \omega\,\theta. \tag{35}$$

It exactly matches. In order to then get the solution in terms of p, q one would have to construct the inverse transformation.

Local integrability

Locally one can always find enough first integrals, so that each system basically is locally integrable. This is guaranteed by the following two results

Proposition 2.3 *Consider a smooth function $F : U \to \mathbb{R}^{2n}$ on an open subset of $\mathcal{M}$. If the initial value problem $\dot{x} = F(x)$ with $x(0) = x_0$ has a smooth solution, then there are n independent first integrals of motion in a neighborhood V of x_0.*

Proof Let $x = \phi(t, x_0)$ be the local solution of the initial value problem. Locally we can invert this solution to express the initial conditions as $x_0 = \tilde{\phi}(t, x_0)$. By definition the initial conditions x_0 are constant and independent.

Superintegrability

It is possible for a system to have more independent first integrals. For example a free particle in one dimension has conserved energy and momentum. Systems for which is the case are called superintegrable. A different example is Kepler's motion of planets. It has conserved angular momentum, energy but als the Runge-Lenz vector is conserved. However, these quantities can not be all in involution.

2.3 Examples

Clearly, any one-dimensional example with conserved energy is trivially integrable. To find more interesting examples we have to go to higher dimensions.

Kepler

We now move on to three dimensions and discuss one of the historically first examples of an integrable system; the Kepler two-body problem of planetary motion. In terms of the Cartesian coordinates, the Hamiltonian is simply given by

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \frac{k}{r}, \qquad\qquad r = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{36}$$

The system clearly has rotation invariance and it is easy to show that the angular momentum $J = x \times p$ is conserved. As always, let us take the following conserved quantities

$$H = \frac{1}{2}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right] + V(r), \qquad J^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}, \qquad J_3 = p_\phi. \tag{37}$$

Without loss of generality, let us set $\vec{J} = (0, 0, J_3)$, or in other words, we restrict the motion to the plane where $\theta = \frac{\pi}{2}$. In this plane we find

$$\dot{\phi} = \{H, \phi\} = \frac{p_\phi}{r^2 \sin^2\theta}, \qquad \Rightarrow \qquad p_\theta = r^2 \dot{\phi}. \tag{38}$$

The is the conservation of angular momentum discovered by Kepler.

We can now construct the solution by following the proof form Liouville's theorem. The momenta p_r, p_ϕ are easily solved in terms of the energy E and angular momentum J

$$p_r = \sqrt{2(E - V) - \frac{J^2}{r^2}}, \qquad\qquad p_\phi = J_3 \equiv J. \tag{39}$$

We then use those to construct the generating function S

$$S = \int_r dr \sqrt{2(H - V) - \frac{J^2}{r^2}} + J \int_\phi d\phi. \tag{40}$$

Then the associated angle variables are simply

$$\psi_E = \frac{\partial S}{\partial E}, \qquad\qquad \psi_J = \frac{\partial S}{\partial J} \tag{41}$$

and they satisfy the following trivial equations of motion

$$\dot{\psi}_E = 1, \qquad\qquad \dot{\psi}_J = 0, \tag{42}$$

so that $\psi_E = t$. In order to transform this back to our original spherical coordinates we plug in the explicit expression for ψ_E and derive

$$t = \int_r \frac{dr}{\sqrt{2(E - V) - \frac{J^2}{r^2}}}. \tag{43}$$

Similarly we get from the equation of ψ_J that

$$\phi = \int_r \frac{J dr}{r^2 \sqrt{2(E - V) - \frac{J^2}{r^2}}} = \arccos \frac{\frac{J}{r} - \frac{k}{J}}{\sqrt{2E + \frac{k^2}{J^2}}}. \tag{44}$$

To get the familiar formulation in which we can distinguish the different types of orbits, let us define

$$p = \frac{J^2}{k}, \qquad\qquad e = \sqrt{1 + \frac{2EJ^2}{k^2}}, \tag{45}$$

which leads to

$$r = \frac{p}{1 + e \cos\theta}. \tag{46}$$

This equation is called the focal equation of a conic section. When $e < 1$ the conic section is an ellipse. The problem has one more conserved quantitiy; the Runge-Lenz vector

$$R = \vec{v} \times \vec{J} - k\frac{\vec{r}}{r}. \tag{47}$$

Hence, in total there are 5 independent conserved quantities, making this problem superintergable.

The Swinging Atwood machine

Consider a system where two masses M, m are connected by a string and move on pulleys. The kinetic energy is simply given by

$$T = \tfrac{1}{2}(m + M)\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 \tag{48}$$

and the potential energy is the gravitational one

$$V = gr(M - m\cos\theta). \tag{49}$$

Thus we get the Hamiltonian

$$H = \frac{1}{2}\left[\frac{p_r^2}{m + M} + \frac{p_\theta^2}{mr^2}\right] + gr(M - m\cos\theta). \tag{50}$$

It turns out that only in the case $\mu = M/m = 3$ the model is integrable. Let us find the second conserved quantity. First we change coordinates

$$r = \tfrac{1}{2}(\xi^2 + \eta^2), \qquad \theta = 2\arctan\frac{\xi^2 - \eta^2}{2\xi\eta}, \tag{51}$$

giving for $\mu = 3$

$$H = \frac{\frac{1}{8}(p_\xi^2 + p_\eta^2) + 2g(\xi^4 + \eta^4)}{\xi^2 + \eta^2}. \tag{52}$$

Thus for any value of the energy $H = E$ this allows us to write

$$(p_\xi^2 + 16g\,\xi^4 - 8E\,\xi^2) = -(p_\eta^2 + 16g\,\eta^4 - 8E\,\eta^2) = F = const. \tag{53}$$

It can be shown that this system is only integrable for this particular mass ratio.

3 Action-angle variables

As we have shown in the Liouville theorem on integrability, the phase space for integrable systems is foliated in tori on which the motion takes place. In particular if the phase space is $2n$ dimensional, then the motion happens on an n-torus defined as the level surface of the n commuting integrals of motion.

There is a natural set of coordinates (F, ψ) which are called action-angle coordinates. The n-torus has n fundamental cycles C_m, for which we can define constants of motion in a way similar to the generating function of Liouville's theorem

$$I_m = \frac{1}{2\pi} \oint_{C_m} \alpha. \tag{54}$$

Clearly I_m are functions of f_i only and hence constants of motion. Furthermore, they have the dimension of an action and are therefore called action variables.

Now consider the generating function $S(I, q)$ and use it to introduce the angle variables and momentum

$$\phi_i = \frac{\partial S}{\partial I_i}, \qquad\qquad p_i = \frac{\partial S}{\partial q_i} \tag{55}$$

It is quickly seen that the coordinate transformation to action-angle variables is canonical. The angle variables ϕ_i nicely parameterize the circles of the torus in the sense that

$$\frac{1}{2\pi} \int_{C_i} d\phi_j = \delta_{ij}. \tag{56}$$

Indeed

$$\begin{aligned}
\frac{1}{2\pi} \int_{C_i} d\phi_j &= \frac{1}{2\pi} \frac{\partial}{\partial I_j} \int_{C_i} dS \\
&= \frac{1}{2\pi} \frac{\partial}{\partial I_j} \int_{C_i} \left[\frac{\partial S}{\partial q_i} dq_i + \frac{\partial S}{\partial I_i} dI_i \right] \\
&= \frac{1}{2\pi} \frac{\partial}{\partial I_j} \int_{C_i} \alpha \\
&= \frac{\partial}{\partial I_j} I_i = \delta_{ij},
\end{aligned} \tag{57}$$

where we used that the last part of the integral in step two vanishes on the contour.

Notice also that the equations of motion in terms of the action-angle variables are once again trivial

$$\dot{I}_n = 0, \qquad\qquad \dot{\phi}_n = \omega_n = \frac{\partial H}{\partial I_n}. \tag{58}$$

Action-angle variables play an important role in perturbation theory.

Harmonic Oscillator The phase space was simply divided in a set of ellipses in the (q, p) described by

$$p = \pm\sqrt{2E - \omega^2 q^2}, \tag{59}$$

and therefore, the action variable is given by

$$I = E/\omega. \tag{60}$$

Then, via the generating function we find the angle variable

$$\theta = \omega \arctan \frac{q}{\sqrt{2I - q^2}} \qquad \Rightarrow \qquad q = \sqrt{2I} \sin \frac{\theta}{\omega}. \tag{61}$$

We see that we recover our original solution.

4 Lax pairs and r-matrix

A more modern approach to integrable system is formulated in terms of Lax pairs. Consider two matrices L, M whose entries are functions on the phase space $\mathcal{M}$. Suppose the equations of motions can be written as

$$\dot{L} = [L, M], \tag{62}$$

then L, M are called a Lax pair for the system. A very important property is that L automatically generates conserved quantities

$$F_k = \mathrm{tr} L^k. \tag{63}$$

This follows trivially from the equations of motion

$$\dot{F}_k = k\,\mathrm{tr}(L^{k-1}[L, M]) = 0, \tag{64}$$

by cyclicity of the trace. This directly implies that the Lax matrix L is isospectral, *i.e.* its eigenvalues are constant. In particular, this allows us to solve the time dependence explicitly

$$L(t) = g(t)L(0)g(t)^{-1}, \qquad M = \dot{g}(t)g^{-1}(t), \tag{65}$$

which follows directly from the Lax equation.

Similarly, it is easy to see that a Lax pair is not unique. In fact, for any invertible matrix g, it is easy to show that

$$L' = gLg^{-1}, \qquad M' = gMg^{-1} + \dot{g}g^{-1} \tag{66}$$

also defines a Lax pair. Moreover, for any matrix $\tilde{M}$ such that $[L, M - \tilde{M}] = 0$ we find that $L, \tilde{M}$ also forms a Lax pair. Any power L^k of L also satisfies the Lax equation.

Let us look at our favorite example, the harmonic oscillator. We can define the Lax pair as

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & -\frac{1}{2}\omega \\ \frac{1}{2}\omega & 0 \end{pmatrix}. \tag{67}$$

Then the Lax equation gives

$$\begin{pmatrix} \dot{p} & \omega\dot{q} \\ \omega\dot{q} & -\dot{p} \end{pmatrix} = \begin{pmatrix} -\omega^2 q & \omega p \\ \omega p & \omega^2 q \end{pmatrix}. \tag{68}$$

We find the Hamiltonian as $H = \frac{1}{4}\mathrm{tr} L^2$. It is also very easy to factor out the time dependence. Solving the equation for g gives

$$\dot{g} = Mg, \qquad \Rightarrow \qquad g = \begin{pmatrix} \cos\frac{\omega t}{2} & -\sin\frac{\omega t}{2} \\ \sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{pmatrix}, \tag{69}$$

while the eigenvalues of L are simply $\pm\sqrt{E}$.

The Toda system Toda derived a simple model for the dynamics of interacting particles with repulsive exponential forces. We have the Hamiltonian

$$H = \sum_{i=1}^{N} \left[\tfrac{1}{2}p_i^2 + X_i\right], \qquad X_i = e^{q_i - q_{i+1}} \tag{70}$$

The term q_{N+1} is not present. The Hamiltonian equations of motion are simply

$$\dot{q}_i = p_i, \qquad \dot{p}_i = X_i - X_{i-1}, \qquad \dot{p}_1 = X_1, \qquad \dot{p}_N = -X_{N-1}. \tag{71}$$

Now define

$$a_i = \tfrac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})}, \qquad b_i = -\tfrac{1}{2} p_i, \tag{72}$$

then

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 & 0 \\ a_1 & b_2 & a_2 & \dots & 0 & 0 \\ 0 & a_2 & b_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{N-1} & b_N \end{pmatrix}, \quad M = \begin{pmatrix} 0 & a_1 & 0 & \dots & 0 & 0 \\ -a_1 & 0 & a_2 & \dots & 0 & 0 \\ 0 & -a_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -a_{N-1} & 0 \end{pmatrix} \tag{73}$$

is a Lax pair for the open Toda model. Let us restrict to the case where $N = 3$. Then we get the conserved quantities as generated by the Lax pair

$$I_1 = H \tag{74}$$

$$I_2 = p_1 + p_2 + p_3 \tag{75}$$

$$I_3 = \tfrac{1}{3}(p_1^3 + p_2^3 + p_3^3) + p_1 X_1 + p_2(X_1 + X_2) + p_3 X_2. \tag{76}$$

The first and second correspond to total energy and total momentum.

Furthermore, any integrable system admits a (trivial) Lax pair. To see this, we work in the action-angle variables (I, ϕ) such that

$$\dot{I}_k = 0, \qquad \dot{\phi}_k = \frac{\partial H}{\partial I_k} = \omega_k. \tag{77}$$

Then we can define the Lax pair

$$L = \mathrm{diag}(A_1, A_2, \dots), \qquad M = \mathrm{diag}(B_1, B_2, \dots) \tag{78}$$

where

$$A_k = \begin{pmatrix} I_k & 2I\phi \\ 0 & -I_k \end{pmatrix}, \qquad B_k = \begin{pmatrix} 0 & \omega_k \\ 0 & 0 \end{pmatrix}. \tag{79}$$

This construction of a Lax pair however is rather useless since it requires the action-angle variables explicitly. And of course, if these are known, there is no need for a Lax pair.

Lax pairs depending on spectral parameters

There is an important further generalization of a Lax pair, where both L, M depend on an additional parameter λ, called the spectral parameter. The first integrals defined in the previous section now encompass more conserved quantities

$$I_k(\lambda) = \mathrm{tr} L^k(\lambda) = \sum_i I_{k,i} \lambda^i. \tag{80}$$

In particular each of the coefficients $I_{k,i}$ is trivially a first integral. In a sense it offers a more compact way of writing down a Lax pair. The following proposition tells us that such a Lax pair is equivalent to a Lax pair without spectral parameter

Proposition 4.1 *Let $L(\lambda), M(\lambda)$ be a Lax pair depending on a spectral parameter. Suppose we can write $L = \sum_{i=1}^{a} L_i \lambda^i$ and $M = \sum_{i=1}^{b} M_i \lambda^i$, then*

$$\tilde{L} = \begin{pmatrix} L_0 & L_1 & \dots & L_a & 0 & \dots & 0 & 0 \\ 0 & L_0 & L_1 & \dots & L_a & \dots & 0 & 0 \\ 0 & 0 & L_0 & L_1 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \dots & L_0 & L_1 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & L_0 \end{pmatrix}, \tag{81}$$

$$\tilde{M} = \begin{pmatrix} M_0 & M_1 & \dots & M_b & 0 & \dots & 0 & 0 \\ 0 & M_0 & M_1 & \dots & M_b & \dots & 0 & 0 \\ 0 & 0 & M_0 & M_1 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \dots & M_0 & M_1 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & M_0 \end{pmatrix}, \tag{82}$$

is a Lax pair. The conserved quantities are now generated by summing along the diagonals in the upper triangular part.

Proof Expanding the ordinary Lax equation in powers of λ gives

$$\sum_{i=0}^{a} \dot{L}_i \lambda^i = \sum_{i=0}^{a} \sum_{j=0}^{b} [L_i, M_j] \lambda^{i+j}. \tag{83}$$

We can then simply compare order by order, which gives

$$\dot{L}_i = \sum_{j=0}^{i} [L_j, M_{i-j}], \tag{84}$$

where $L_j = M_k = 0$ if $j > a, k > b$. It is now easily shown that these are exactly the relations obtained from the Lax pair $\tilde{L}, \tilde{M}$. Moreover, the conserved quantities

$$I_1(\lambda) = \mathrm{tr} L = \lambda^i \mathrm{tr} L_i \tag{85}$$

$$I_2(\lambda) = \mathrm{tr} L^2 = L_0^2 + \lambda(L_0 L_1 + L_1 L_0) + \lambda^2 (L_1^2 + L_0 L_2 + L_2 L_0) + \dots \tag{86}$$

trivially follow from summing along the diagonals of $\tilde{L}, \tilde{L}^2, \dots$.

The above proposition shows that any λ-dependent Lax pair is equivalent to an ordinary Lax pair, but of rather large size. In the context of field theories, where the number of degrees of freedom tends to infinity, this of course will no longer work.

Kepler Let us return to the Kepler problem. In this case, Lax pair depends on a spectral parameter λ and three additional parameters λ_k. It is given by

$$L = \frac{1}{2} \sum_k \begin{pmatrix} -\frac{x_k \dot{x}_k}{\lambda - \lambda_k} & \frac{x_k x_k}{\lambda - \lambda_k} \\ -\frac{\dot{x}_k \dot{x}_k}{\lambda - \lambda_k} & \frac{x_k \dot{x}_k}{\lambda - \lambda_k} \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & -1 \\ \frac{k}{r^3} & 0 \end{pmatrix} \tag{87}$$

The equations of motion can be read off by expanding around the poles λ_i. The components of the angular momentum are generated by L^2.

Generating integrable systems

We can generate al new integrable systems from an existing Lax pair. Let (L, M) be a Lax pair. Consider the generalized Lax equation

$$\dot{L} = [L, M] + \lambda L, \tag{88}$$

where λ is some constant. Then the corresponding set of equations of motion define an integrable system with first integrals $I_k = \mathrm{tr}(L^k)e^{-k\lambda t}$. The resulting system in general does not come from a Hamiltonian.

Harmonic oscillator The equations of motion corresponding to the generalized Lax equation become

$$\dot{q} = p + \lambda q, \qquad \dot{p} = -\omega^2 q + \lambda p. \tag{89}$$

This system of equations do not come from a Hamiltonian. Nevertheless, we can simply solve them and get

$$q = e^{t\lambda}(A\cos\omega t + B\sin\omega t) \tag{90}$$

and we find that $I_2 = e^{-2\lambda t}(p^2 + \omega^2 q^2)$ is a first integral.

The r-matrix

So far we have not made any mention of the Poisson structure. But of course, for a Lax pair to define an integrable system, the eigenvalues of L need to be in involution. This property turns out to be equivalent to the existence of a so-called (classical) r-matrix.

Suppose we are given a Lax pair L, M, which are $N \times N$ matrices and that the Lax matrix L is diagonalizable

$$L = U\Lambda U^{-1}. \tag{91}$$

The matrix elements λ_k of the diagonal matrix Λ are conserved quantities. We will not consider the question of independence of these quantities.

Let E^i be the standard N basis vectors and let E^i_j be the standard matrix unities, such that $E^i_j E^k = \delta^k_j E^i$, then we can write our Lax matrix as

$$L = L^i_j E^j_i, \tag{92}$$

where L^i_j are simply the entries of the Lax matrix, which are functions on phase space. We want to be able to compute Poisson brackets between those entries. In other words, we want to evaluate $\{L^i_j, L^k_l\}$. Let us embed the Lax matrix in the double tensor product

$$L_1 \equiv L \otimes 1 = L^i_j E^j_i \otimes 1, \qquad L_2 \equiv 1 \otimes L = L^i_j 1 \otimes E^j_i. \tag{93}$$

Thus, the subscript refers to the space the matrix is embedded in. Similarly, let us consider matrices that act on the tensor product

$$T_{12} = T^{ij}_{kl} E^k_i \otimes E^l_j, \qquad T_{21} = T^{ij}_{kl} E^l_j \otimes E^k_i. \tag{94}$$

Then T_{21} is simply the permuted version of T. For any such matrix we can also take the partial trace over one of the factors in the tensor product. For example, we have

$$\mathrm{tr}_1 T_{12} = T^{ij}_{kl}\mathrm{tr}(E^k_i)E^l_j = T^{ij}_{il}E^l_j. \tag{95}$$

We can now conveniently put all the possible Poisson brackets between the entries of L into the following matrix form $\{L_1, L_2\}$. If the eigenvalues of the Lax matrix are in involution, then the latter Poisson bracket needs to be of a special form.

Proposition 4.2 *The eigenvalues of the Lax matrix are in involution if and only if there exists a function r_{12} on the space space such that*

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]. \tag{96}$$

Proof Let us first prove that if (96) holds the eigenvalues Poisson commute. By the Leibniz rule of the Poisson bracket we have that

$$\{L_1^n, L_2^m\} = [a^{n,m}_{12}, L_1] + [b^{n,m}_{12}, L_2], \tag{97}$$

with

$$a^{n,m}_{12} = \sum_{p=0}^{n-1}\sum_{q=0}^{m-1} L_1^{n-p-1}L_2^{m-q-1}r_{12}L_1^pL_2^q \tag{98}$$

$$b^{n,m}_{12} = -\sum_{p=0}^{n-1}\sum_{q=0}^{m-1} L_1^{n-p-1}L_2^{m-q-1}r_{21}L_1^pL_2^q. \tag{99}$$

Then we take trace of (97) and use the the trace of a commutator vanishes to obtain that the quantities $\mathrm{tr}L^n$ are in involution. This is equivalent to the eigenvalues Poisson commuting.

On the other hand, by the Leibniz rule we can completely expand the Poisson bracket of the matrix entries of L into nine terms

$$\begin{aligned}\{L_1, L_2\} =&\{U_1\Lambda_1U_1^{-1}, U_2\Lambda_2U_2^{-1}\} \\ =&\{U_1, U_2\}\Lambda_1U_1^{-1}\Lambda_2U_2^{-1} + U_1\{\Lambda_1, U_2\}U_1^{-1}\Lambda_2U_2^{-1} - L_1\{U_1, U_2\}U_1^{-1}\Lambda_2U_2^{-1}+ \\ &U_2\{U_1, \Lambda_2\}\Lambda_1U_1^{-1}U_2^{-1} + U_1U_2\{\Lambda_1, \Lambda_2\}U_1^{-1}U_2^{-1} - L_1U_2\{U_1, \Lambda_2\}U_1^{-1}U_2^{-1}- \\ &L_2\{U_1, U_2\}U_2^{-1}\Lambda_1U_1^{-1} - L_2U_1\{\Lambda_1, U_2\}U_1^{-1}U_2^{-1} + L_1L_2\{U_1, U_2\}U_1^{-1}U_2^{-1},\end{aligned} \tag{100}$$

where we used that $\{U^{-1}, A\} = U^{-1}\{U, A\}U^{-1}$. Using that $\{\Lambda_1, \Lambda_2\}$ vanishes and introducing the following quantities

$$k_{12} = \{U_1, U_2\}U_1^{-1}U_2^{-1}, \qquad q_{12} = U_2\{U_1, \Lambda_2\}U_1^{-1}U_2^{-1}, \tag{101}$$

we can write

$$\{L_1, L_2\} = k_{12}L_1L_2 + L_1L_2k_{12} - L_1k_{12}L_2 - L_2k_{12}L_1 - [q_{21}, L_2] + [q_{12}, L_1]. \tag{102}$$

Then if we define

$$r_{12} = q_{12} + \tfrac{1}{2}[k_{12}, L_2], \tag{103}$$

we can finally bring the commutation relation to the required form

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]. \tag{104}$$

The Poisson bracket should satisfies the Jacobi identity and this of course puts some further restrictions on the r-matrix. Namely, it should satisfy

$$\{L_1, [r_{12}, r_{13}] + [r_{12}, r_{23}] + \{L_2, r_{13}\} - \{L_3, r_1\}\} + \text{cycl.perm.} = 0. \tag{105}$$

If the r-matrix is constant, then only the first term survives and the Jacobi identity is satisfied if

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{23}, r_{13}] = 0. \tag{106}$$

If the r-matrix is antisymmetric $r_{12} = -r_{21}$ then the corresponding equation is called the classical Yang-Baxter equation. Thus, even though the above proof of the existence of an r-matrix is constructive, we are only dealing with an integrable system if and only if (105) is satisfied. In a sense, classifying classical integrable systems corresponds to classifying solutions of this equation.

The fact that the eigenvalues of the Lax matrix commute is equivalent to the existence of an r-matrix. Since the eigenvalues of the Lax matrix mutually Poisson commute, the quantities $I_n = \text{tr}(L^n)$ do as well. We can now show that the time evolution with respect to any of these first integrals is naturally of Lax form.

Proposition 4.3 *Let L, r be matrices such that $\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]$. If we take $I_n = \text{tr}(L^n)$ as conserved quantities, then the equations of motion admit a Lax representation*

$$\frac{dL}{dt_n} \equiv \{I_n, L\} = [M_n, L], \qquad M_n = -n\text{tr}_1(L_1^{n-1} r_{21}). \tag{107}$$

Proof From the proof of the existence of the r-matrix we had the equation

$$\{L_1^n, L_2^m\} = [a_{12}^{n,m}, L_1] + [b_{12}^{n,m}, L_2]. \tag{108}$$

Setting $m = 1$ and taking the trace over the first space proves the proposition.

It is easy to see that the r-matrix is not unique. For example you can simply add the identity matrix to it. Furthermore, the structure of the Poisson brackets is preserved by the gauge transformation $L \to L' = gLg^{-1}$, such that

$$\{L_1', L_2'\} = [r_{12}', L_1'] - [r_{21}', L_2'], \tag{109}$$

where

$$r_{12}' = g_1 g_2 (r_{12} + g_1^{-1}\{g_1, L_2\} + \tfrac{1}{2}[g_1^{-1} g_2^{-1}\{g_1, g_2\}, L_2]) g_1^{-1} g_2^{-1}. \tag{110}$$

In other words, gauge transformation leave the involution property invariant. We finish this chapter with two examples; the Harmonic oscillator and the open Toda chain.

Harmonic Oscillator The r-matrix is most easily computed by using the variables ρ, θ such that

$$p = \rho \cos\theta, \qquad q = \frac{\rho}{\omega} \sin\theta. \tag{111}$$

These variables are not canonical since $\{\rho, \theta\} = \omega/\rho$. Then L is diagonalized by

$$U = \begin{pmatrix} \cos\frac{1}{2}\theta & \sin\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta & -\cos\frac{1}{2}\theta \end{pmatrix}. \tag{112}$$

Since U depends solely on θ the commutator $\{U_1, U_2\} = 0$ and our general expression for the r-matrix simplifies a lot and we find

$$r = \frac{\omega}{2\rho^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes L. \tag{113}$$

This r-matrix is a so-called dynamical r-matrix since it depends on the dynamical variables, *i.e.* it is a non-trivial function on phase space.

Open Toda model The Lax matrix for the open Toda model with N particles can be written as

$$L = \sum_i b_i E_i^i + a_i(E_{i+1}^i + E_i^{i+1}), \tag{114}$$

where E_j^i are again the matrix unities and a, b are defined in (72). The r-matrix here is constant and antisymmetric and is of the form

$$r = \frac{1}{4}\sum_i (E_{i+1}^i \otimes E_i^{i+1} - E_i^{i+1} \otimes E_{i+1}^i). \tag{115}$$

You can find r through direct computation. It is readily checked that this r-matrix satsifies the classical Yang-Baxter equation.

5 Spectral curve

There is a close connection between integrable systems that admit a Lax pair representation depending on a spectral parameter and Riemann surfaces. Sometimes this formulation can help in solving the integrable model, but in general it simply gives an interesting relation between two seemingly unrelated topics.

Riemann surfaces

A Riemann surface S is a one-dimensional smooth compact complex manifold. This means that around each point $p \in S$ there is an open set U that is diffeomorphic to the open disc around the origin. And in each overlapping open subset the different local parameters are related by an analytic bijection.

We will be interested in the algebraic formulation, where we define a Riemann surface via an algebraic equation in $\mathbb{C}^2$. So we consider a polynomial $P(x, y)$ in two variables and define the Riemann surface as

$$S = \{(x, y) \in \mathbb{C}^2 | P(x, y) = 0\}. \tag{116}$$

Since we want it to be compact, we also add the point at infinity. We map it to a chart around 0 by transforming $x \to x^{-1}, y \to y^{-1}$.

Suppose P has degree N in y, then generically above any point x there will be N solutions to the defining equations. Thus, we get an N-fold covering of the complex line. Then there will be some branch points where the solutions have higher multiplicity.

A defining topological quantity of a Riemann surface is the genus. A surface of genus g is homeomorphic to a sphere with g handles. So a sphere has genus 0 and a torus has genus 1. For a curve defined by an algebraic equation you can compute the genus by the Riemann-Hurwitz formula, which relates the genus to the degree of covering and the number of branch cuts.

We will be considering meromorphic functions on S. Finally, let us introduce the notion of a divisor. A divisor is simply a formal sum of points with multiplicities and we simply denote it by $D = \sum_j n_j P_j$. For any meromorphic function f with zeroes at position p_j with multiplicity n_j and poles at q_j with multiplicty m_j we define its divisor as

$$D(f) = \sum n_j p_j - \sum m_j q_j. \tag{117}$$

A divisor is called positive if all n are positive and m vanish. Hence a function is analytic if and only if its divisor is positive. The so-called Riemann-Roch theorem states how big the vector space of functions with a certain divisor is.

The Spectral curve

Consider an integrable system with Lax pair $L(\lambda), M(\lambda)$. Then we can define a Riemann surface via the locus of the characteristic equation

$$\Gamma : \det(L - \mu) = 0. \tag{118}$$

This defines a Riemann surface in $\mathbb{C}^2$. If the Lax matrix is of size $N \times N$, then the equation is of the form

$$(-\mu)^N + \sum r_i(\lambda, p, q)\mu^i = 0. \tag{119}$$

Hence this defines an N-sheeted cover of the Riemann sphere. To any given base point λ there correspond N values μ_i corresponding to the different eigenvalues of L at that particular value of the spectral parameter.

We will now try to find out the analytical properties of the eigenvectors and the Lax pair. In fact we will try to determine how the full information of the underlying integrable system is realized on the spectral curve.

First remember that the Lax matrix was more or less defined up to gauge transformations. However, the spectral curve is clearly invariant under such tranformations. Furthermore, since the eigenvalues are time-independent, the spectral curve is as well. Thus we immediately see that the curve itself does not contain any dynamical information. Before we go any further, let us fix the gauge on the Lax matrix by requiring that at $\lambda = \infty$ it is diagonal

$$L(\infty) = \mathrm{diag}(a_1, \dots, a_N). \tag{120}$$

We will only work in the generic setting where all the a_i are different.

A matrix is fully specified by giving its eigenvalues and eigenvectors. We have seen that the eigenvalues determine the spectral curve, so let us now consider the eigenvectors. To each point $P = (\lambda, \mu)$ on the curve we can assign the corresponding eigenvector

$$\psi(P) = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}. \tag{121}$$

Eigenvectors are obviously defined up to a normalization, so we are free to pick up a convenient one. For now we set $\psi_1 = 1$. Since eigenvectors can be obtained by taking suitable minors of the matrix $L - \mu$ we directly see that they are meromorphic functions on the spectral curve.

Let us now look at the pole structure of the eigenvectors. We say that the eigenvector has a pole if and only if one of its components has a pole. We have N points at $\lambda = \infty$ and since the matrix diagonalizes there, we simply find that the eigenvectors correspond to the standard basis vectors. However, when we pick up the normalization $\psi_1 = 1$ this gives a problem. Obviously at $P_1 = (\infty, a_1)$

$$\psi(P_1) = (1, \mathcal{O}(\lambda^{-1}), \dots, \mathcal{O}(\lambda^{-1})), \tag{122}$$

but at other points P_i we need to have

$$\psi(P_i) = (1, \mathcal{O}(\lambda^{-1}), \dots, \mathcal{O}(\lambda), \dots, \mathcal{O}(\lambda^{-1})), \tag{123}$$

for it to be an eigenvector. Thus we find $N-1$ poles at $\lambda = \infty$. These poles are completely fixed by the asymptotic behaviour of the Lax matrix and consequently are also not dynamical.

In fact, one can show on general grounds that the total number of poles of the eigenvector ψ is related to the genus of the spectral curve and dimension N as

$$\#poles = g + N - 1. \tag{124}$$

Having found the $N-1$ poles at infinity, we are only let with g poles on the curve. These contain all the dynamical information and putting them together in a divisor, we get the so-called dynamical divisor D. Now surprisingly, we can revert the logic. There is the Riemann-Roch theorem that implies that for a given divisor D of degree g, there is a unique meromorphic function that has exactly the analytic properties of the eigenvector ψ.

Let us take a step back and think about what this teaches us. The full data of our integrable model is divided into two parts. We have the conserved quantities (action variables) that define the spectral curve and then we basically need to specify the position of g poles (angle variables) and that parameterizes the dynamical data of the theory. In fact this construction proved useful in developing the method of separation of variables.

If we assume some definite form of our Lax pair L, M we can make the dynamical nature of D more explicit. From the Lax equation we find that the time evolution of the eigenvector is described by

$$(L - \mu)(\dot{\psi} - M\psi) = 0. \tag{125}$$

In other words we have

$$\dot{\psi}(t, P) = (M(\lambda) - c(t, P))\psi, \tag{126}$$

where c is some scalar function. If we want to keep the first component of the eigenvector always fixed to be 1, then we get that $c = M_{1j}\psi_j$. From this you can fully fix the time evolution of the dynamical divisor. The initial positions of the poles is fully fixed by the initial conditions of the system and thus the system can be solved in this way.

The closed Toda chain

Let us try to clarify all the notions a bit by considering an example. Earlier we discussed the open Toda chain, but its Lax matrix did not depend on a spectral parameter. However, if we put the particles on a circle rather than a line, we do find that the Lax matrix becomes of spectral form.

Model. We view $n+1$ particles sitting on a circle with an potential that exponentially depends on the relative distances. The Hamiltonian is

$$H = \sum_{i=1}^{N} \left[\tfrac{1}{2}p_i^2 + X_i\right], \qquad X_i = e^{q_i - q_{i+1}} \tag{127}$$

and the corresponding equations of motion are

$$\dot{q}_i = p_i, \qquad \dot{p}_i = e^{q_{i-1}-q_i} - e^{q_i - q_{i+1}}, \tag{128}$$

where it is understood that $(p_a, q_a) \equiv (p_{a+n+1}, q_{a+n+1})$. Notice that the system has translation symmetry $q_i \to q_i + a$. We impose that the centre of mass is standing still at the origin, *i.e.* we set

$$\sum_i p_i = \sum_i q_i = 0. \tag{129}$$

This leaves us with $2n$ degrees of freedom and hence our phase space is $2n$ dimensional. The Lax pair is this time given by

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 & \lambda^{-1}a_{n+1} \\ a_1 & b_2 & a_2 & \dots & 0 & 0 \\ 0 & a_2 & b_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda a_{n+1} & 0 & 0 & \dots & a_n & b_{n+1} \end{pmatrix},$$
$$M = \begin{pmatrix} 0 & a_1 & 0 & \dots & 0 & -\lambda^{-1}a_{n+1} \\ -a_1 & 0 & a_2 & \dots & 0 & 0 \\ 0 & -a_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda a_{n+1} & 0 & 0 & \dots & -a_n & 0 \end{pmatrix} \tag{130}$$

It is easy to check that the equations of motion are equivalent to the Lax equation $\dot{L} = [L, M]$. To show that this is indeed an integrable model, we also need an r-matrix. It is given by

$$r = \frac{1}{2}\frac{\lambda + \lambda'}{\lambda - \lambda'} + \frac{1}{\lambda - \lambda'}\sum_{i<j}(\lambda' E_{ij} \otimes E_{ji} + \lambda E_{ji} \otimes E_{ij}). \tag{131}$$

Spectral curve. Let us now construct the spectral curve Γ coming corresponding to the closed Toda model. The spectral curve is defined by the equation

$$\det(L - \mu) = 0. \tag{132}$$

For fixed λ this is simply the characteristic equations describing the eigenvalues of L. For concreteness, let us set $n = 2$. The following can be easily generalized to work for general n.

From the explicit expression for our Lax matrix we can easily find the algebraic equation that describes our spectral curve

$$\lambda + \lambda^{-1} = -\mu^3 + \mu\left[\sum_i a_i^2 - \frac{b_1 b_2 b_3}{b_i}\right] + \left[b_1 b_2 b_3 - (a_1^2 b_3 + a_2^2 b_1 + a_3^2 b_2)\right]. \tag{133}$$

This is a so-called hyperelliptic curve. The genus of this elliptic curve is 2 and hence it simply is a double torus.

Now let us look at the pole structure. The Lax matrix has two poles, namely at $\lambda = 0, \infty$. Around these poles, we can expand the defining equation. We see that

$$\lambda = \mu^3(1 - \mu^{-1} \sum_i p_i + \mathcal{O}(\mu^{-2}), \qquad \lambda \to \infty \tag{134}$$

$$\lambda = \mu^{-3}(1 + \mu^{-1} \sum_i p_i + \mathcal{O}(\mu^{-2}), \qquad \lambda \to 0. \tag{135}$$

Eigenvectors. Let us start by writing down the explicit eigenvalue equations corresponding to L. Let $\Psi = (\psi_1, \psi_2, \psi_3)$ be the eigenvector corresponding to eigenvalue μ then

$$\begin{aligned} p_1\psi_1 + a_1\psi_2 + \lambda^{-1}a_3\psi_3 &= \mu\psi_1, \\ p_2\psi_2 + a_1\psi_1 + a_2\psi_3 &= \mu\psi_2, \\ p_3\psi_3 + a_2\psi_2 + \lambda a_3\psi_1 &= \mu\psi_3. \end{aligned} \tag{136}$$

With the help of mathematica we can compute the eigenvectors explicitly. It obviously corresponds to a cubic equation. But nevertheless, we can numerically examine the pole structure. If we fix some generic eigenvalues and normalize the eigenvectors appropriately, it is easily seen that we find 2 additional poles as we should.

At $\lambda = \infty$ we can be more precise. Let us normalize the eigenvectors such that $\psi_3 = \lambda$. From our general discussion we should find 2 poles there. We can solve the eigenvector equations perturbatively to highest order in λ, μ and find

$$\psi_i = e^{q_i - q_0}\mu^i(1 - \mu^{-1}\sum_{j=0}^{i-1} p_j + \mathcal{O}(\mu^{-2})), \qquad \lambda \to \infty \tag{137}$$

$$\psi_i = e^{-q_i + q_0}\mu^{-i}(1 + \mu^{-1}\sum_{j=0}^{i-1} p_j + \mathcal{O}(\mu^{-2})), \qquad \lambda \to 0. \tag{138}$$

Thus we indeed find a pole of order 2 at infinity.

From the Lax equation we can also find the explicit time dependence of the eigenvectors at infinity. We can write

$$\psi_i(t) = e^{q_i(t)}e^{-\mu t}\mu^i(1 + \mathcal{O}(\mu^{-1})), \qquad \lambda \to \infty \tag{139}$$

$$\psi_i(t) = e^{-q_i(t)}e^{\mu t}\mu^{-i}(1 + \mathcal{O}(\mu^{-1})), \qquad \lambda \to 0. \tag{140}$$

Finally one can solve the exact time dependence of the eigenvector by using the complex structure. The result is expressed in so-called tau-functions and can be found in the book by Babelon,Bernard and Talon.