# 1 Introduction

The next three lectures will be about quantum integrable systems that were already touched upon previously. We will use the most simple example (no it's not the Harmonic oscillator) to illustrate the various important notions. This model is the so-called Heisenberg spin chain. Even though an exact definition of what quantum integrability is remains a bit sketchy, we will adopt the working definition that was introduced last week. There should be some underlying monodromy matrix that generates enough conserved quantities, or in the language of classical mechanics; the system admits a Lax pair.

For classical mechanics, Lax pairs are in a sense a modern approach to study and solve them. Of course, as we saw we could already find solutions by using Liouville's theorem. Something similar can be done for integrable spin chains. In quantum mechanics you are interested in finding the energy spectrum and the eigenvectors of the Hamiltonian. True to their classical counterparts, for these integrable systems we can exactly describe those. Thus, they are solvable.

In the first lecture we will start with this 'basic' way of looking at the integrable Heisenberg spin chain. We will simply give a Hamiltonian and solve for its spectrum by a technique called the Bethe Ansatz. This is a very general technique that is used to describe the spectrum of quantum integrable systems.

After this we make contact with the previous lecture and introduce a monodromy, transfer and R-matrix. These can be again used to solve the model by using a modified version of the Bethe Ansatz. Furthermore, this will generate the tower of conserved charges that are crucial in integrable systems.

As you have noticed in the last couple of weeks, integrable systems are closely related to a system having a large amount of conserved charges. As you all know, by Noether's theorem, these usually have their origin in symmetries. The aim of the last lecture will be to study this relation in more detail by looking at the Heisenberg spin chain. We will see that there is an infinite dimensional algebra, closely related to a Lie algebra that appears. In fact this symmetry algebra allows us to derive the monodromy and R-matrix.

# 2 The Heisenberg spin chain

The main model we will study the next three weeks is the so-called Heisenberg spin chain. This is a one-dimensional model of magnetism or simply of spin-$\frac{1}{2}$ particles that have a spin-spin interaction. Actually, this is not just a nice toy model. In some metals and crystals where this is some one-dimensional isotropy these spin chain actually appear and describe the dominant physical behaviour.

## 2.1 Quantum Spin Chain.

The spin chain simply consists of $N$ sites, where on each site we consider a spin-$\frac{1}{2}$ particle (for example an electron). This electron can have spin up or down and, consequently, any electron is in a linear state $a|\uparrow\rangle + b|\downarrow\rangle$ generating a two-dimensional Hilbert space (local). If we have $N$ electrons then the total Hilbert space in which the physical states live is simply

$$H = \bigotimes_N \mathbb{C}^2. \tag{1}$$

The spin operators $S_i^{x,y,z}$ simply act on each site $i$ and they satisfy local commutation relations in the sense that

$$[S_i^a, S_j^b] = \delta_{ij}\epsilon^{abc}S_i^c, \qquad \text{if } i \neq j. \tag{2}$$

The Hamiltonian describes a nearest neighbor spin-spin interaction. More precisely, we have

$$\mathcal{H} = \frac{JN}{4} - J\sum_i \vec{S}_i \cdot \vec{S}_{i+1}, \qquad \vec{S}_{N+1} = \vec{S}_1. \tag{3}$$

Let us introduce the usual raising and lowering operators $S^{\pm} = S^x \pm iS^y$, such that

$$\begin{aligned} &S^+|\uparrow\rangle = 0, && S^-|\uparrow\rangle = |\downarrow\rangle, && S^z|\uparrow\rangle = \tfrac{1}{2}|\uparrow\rangle \\ &S^+|\downarrow\rangle = |\uparrow\rangle, && S^-|\downarrow\rangle = 0, && S^z|\downarrow\rangle = -\tfrac{1}{2}|\downarrow\rangle. \end{aligned} \tag{4}$$

Then we can rewrite the Hamiltonian as

$$\mathcal{H} = \frac{JN}{4} - J\sum_i \tfrac{1}{2}(S_i^+S_{i+1}^- + S_i^-S_{i+1}^+) + S_i^zS_{i+1}^z. \tag{5}$$

Let us look at the different terms. The terms involving $S^{\pm}$ are called hopping terms since they move a spin up or spin down to a neighboring site. The constant term proportional to $N$ is added for convenience. It is simply an overall shift of the energy levels. Depending on the sign, you want to align or anti-align the spins. Hence this is a rudimentary model of (ferro)magnetism.

Written out in components, the Hamiltonian is a special case of a more general Hamiltonian which takes the form

$$\mathcal{H} = \sum_i (J_xS_i^xS_{i+1}^x + J_yS_i^yS_{i+1}^y + J_zS_i^zS_{i+1}^z). \tag{6}$$

This model is usually called the XYZ spin chain. In case $J_x = J_y$ it is called the XXZ spin chain and our model is referred to as the XXX spin chain.

Notice that the size of the Hamiltonian is $2^N \times 2^N$ and hence for large systems it is pretty much impossible to diagonalize it by brute force. In order to determine the spectrum you have to think of a different way. Furthermore, to deal with the physically interesting $N \to \infty$ limit, the spectrum has to be understood with some analytic methods.

## 2.2 Symmetries

One way of reducing the effective size of the Hamiltonian is by looking at symmetries of the system. Consider the operator

$$S^z = \sum_i S_i^z, \tag{7}$$

which measures the total number of up or down spins. It is quickly checked that it commutes with the Hamiltonian. This means that we can restrict to subsets of a fixed number of spins up (or down).

Similarly, you can extend this to all spin operators and define

$$\vec{S} = \sum_i \vec{S}_i \tag{8}$$

and check that they commute with the Hamiltonian. As you all know the spin operators form an $\mathfrak{su}(2)$ algebra and consequently this spin chain has $\mathfrak{su}(2)$ as a symmetry algebra. Actually there is a larger symmetry algebra, but this will be the topic of the last lecture. In any case, this means that the eigenstates of the Hamiltonian will arrange themselves in multiplets with respect to this symmetry algebra.

# 3 Coordinate Bethe Ansatz

The XXX spin chain is exactly solvable in the sense that the spectrum of the Hamiltonian can be described exactly. The 'trick' the makes this possible is to make an Ansatz for the eigenstates of the Hamiltonian. This method is called the coordinate Bethe Ansatz. It was first introduced by H. Bethe in 1931 and it was used exactly for this model. Later this technique has been applied to more general models.

The idea behind the Bethe Ansatz is to consider a reference state (ground state), which is an eigenstate of the Hamiltonian where all the spins are up and then flip some spins. These spins will behave like quasi-particles called magnons.

**Ground state.** Since the total spin is preserved, the state were all spins are pointing in the same direction has to be an eigenstate of the Hamiltonian. This is the ferromagnetic vacuum. Hence, let us define the vacuum to be

$$|0\rangle = |\uparrow\uparrow \dots \uparrow\uparrow\rangle. \tag{9}$$

It is very easy to find the energy of $|0\rangle$

$$\mathcal{H}|0\rangle = 0. \tag{10}$$

We will shortly see that for $J > 0$ this is indeed the ground state of the system.

**Magnons.** The next logical step is to consider the case where some of the spins in the ground state are flipped. The easiest way to flip a spin at position $n$ is by acting with $S_n^-$ on the vacuum. In general we write

$$|n_1, \dots, n_M\rangle = S_{n_1}^- \dots S_{n_M}^- |0\rangle. \tag{11}$$

An eigenstate with $M$ flipped spins is then of the form

$$|\psi\rangle = \sum_{1 \le n_1 < \dots < n_M \le N} a(n_1, \dots, n_M)|n_1, \dots, n_M\rangle, \tag{12}$$

with some unknown coefficients $a(n_1, \dots, n_M)$. Furthermore, periodicity can be formulated as

$$a(n_2, \dots, n_M, n_1 + N) = a(n_1, \dots, n_M) \tag{13}$$

The Bethe Ansatz postulates the form of these coefficients to be

$$a(n_1, \dots, n_M) = \sum_{\sigma \in S_M} A_\sigma e^{ip_{\sigma_i} n_i}. \tag{14}$$

Notice that this is just a plain-wave type Ansatz. We will work this out explicitly for $M = 0, 1, 2$.

**A single magnon.** The next logical step is to consider the case where one of the spins in the ground state is flipped. A priori there are $N$ different states of this type. Let us see how the Hamiltonian acts on such a state

$$\mathcal{H}|\ldots\downarrow_n\ldots\rangle = -J(\tfrac{1}{2}|\ldots\downarrow_{n-1}\ldots\rangle + \tfrac{1}{2}|\ldots\downarrow_{n+1}\ldots\rangle - |\ldots\downarrow_n\ldots\rangle). \tag{15}$$

We see that the flipped spin behaves like a quasi-particle; it can hop between sites or stay put and is called a magnon.

Let's take this analogy a bit further and really consider it as a particle with momentum $p$. Then the most natural way to write the eigenvector simply is

$$|p\rangle = \sum_i e^{ipn} S_n^+ |0\rangle, \tag{16}$$

which is basically the discrete version of a plane-wave. In other words, we take $a(n) = e^{ipn}$. Notice that this is essentially the Fourier transform, which is very natural in the context of a periodic chain.

For the moment let us forget about the periodic boundary conditions and look at a flipped spin at some position $n$. From (15) we see that it also gets a contribution from its neighbouring sites

$$\mathcal{H}|p\rangle = \ldots - \tfrac{1}{2}J\left[e^{ip(n+1)} + e^{ip(n-1)} - e^{ipn}\right]|\ldots\downarrow_n\ldots\rangle\ldots \tag{17}$$

Thus, we see that $|p\rangle$ is an eigenstate of the Hamiltonian and its corresponding energy is

$$E = 2J\sin^2\frac{p}{2}. \tag{18}$$

It is simply a particle with momentum $p$ and energy $E$ moving in the ground state. Notice that the energy is inherently positive and indeed the vacuum is the ground state.

Imposing periodicity should imply a quantization condition on the momentum. What this is can be readily seen by acting with the Hamiltonian on the $N$th site. Instead of getting a contribution from $n = N + 1$ we get a contribution from $n = 1$. Thus we simply find that this state is an eigenstate with eigenvalue $E$ provided that $a(n + N) = a(n)$

$$e^{ipN} = 1. \tag{19}$$

This is the usual momentum quantization condition for a particle on a circle of length $N$.

**Two magnons.** When we have two magnons a new problem arises. Suppose that the chain is very big, then if the two magnons are well-separated we expect them to behave as single magnons since the action of the Hamiltonian is exactly equivalent to the one magnon case. Only when the two excitations are close together the Hamiltonian has a slightly different action.

According to the Bethe Ansatz, the wave function is of the form

$$|p_1, p_2\rangle = \sum_{n_1<n_2}\left[e^{i(p_1n_1+p_2n_2)} + Ae^{i(p_2n_1+p_1n_2)}\right]S_{n_1}^+ S_{n_2}^+|0\rangle. \tag{20}$$

There is a very natural interpretation of this Ansatz by splitting the sum and relabelling the terms as follows

$$|p_1, p_2\rangle = \left[\sum_{n_1>n_2} e^{i(p_1n_1+p_2n_2)} + A(p_1, p_2)\sum_{n_2>n_1} e^{i(p_1n_1+p_2n_2)}\right]S_{n_1}^- S_{n_2}^-|0\rangle. \tag{21}$$

We can view this as two magnons with momenta $p_1, p_2$ at positions $n_1, n_2$. The wave function is divided into two terms; the first magnon to the left of the second one and vice versa. The two different regions are connected byt the constant $A$ which describes the scattering. We will see that basically the magnons pick up a phase when then pass through each other.

Let us now look in detail on how the Hamiltonian acts on this state. Firstly, when $n_1 > n_2+2$ the wave function clearly behaves like two disjoint copies of the one magnon eigenstate. From this we can immediately read off that, if this state is an eigenstate, then the eigenvalue is necessarily given by

$$E(p_1) + E(p_2). \tag{22}$$

This reinforces the idea that we have a couple of almost free particles moving on the spin chain, and their energies simply add up.

The term which displays a new kind of behaviour is when the two magnons are next to each other, say at position $n$ and $n+1$ we have $|\ldots\uparrow\downarrow\downarrow\uparrow\ldots\rangle$. It is easy to see that the only terms that can be mapped to this term by the Hamiltonian are $|\ldots\uparrow\downarrow\downarrow\uparrow\ldots\rangle, |\ldots\downarrow\uparrow\downarrow\uparrow\ldots\rangle$ and $|\ldots\uparrow\downarrow\uparrow\downarrow\ldots\rangle$. From the explicit form of the Bethe ansatz we derive

$$\begin{aligned}\tfrac{1}{2}\left[e^{ip_1(n-1)+ip_2(n+1)} - 2e^{ip_1 n+ip_2(n+1)} + e^{ip_1(n)+ip_2(n+2)}\right] + A\left[p_1 \leftrightarrow p_2\right] = \\ (E(p_1)+E(p_2))(e^{ip_1 n+ip_2(n+1)} + Ae^{ip_2 n+ip_1(n+1)})\end{aligned} \tag{23}$$

From this we can compute $A$ easily. Our two magnon wave function is an eigenstate of the Hamiltonian exactly if

$$A(p_1,p_2) = \frac{\cot\frac{p_1}{2} - \cot\frac{p_2}{2} - 2i}{\cot\frac{p_1}{2} - \cot\frac{p_2}{2} + 2i}. \tag{24}$$

It is easy to see that $A(p_1,p_2) \equiv e^{i\theta_{12}}$ is a pure phase, which is called the scattering matrix (S-matrix). The S-matrix satifies the following condition

$$A(p_1,p_2)A(p_2,p_1) = 1. \tag{25}$$

In our derivation we have assumed that the magnons are not close to the boundary and have again neglected the periodic boundary conditions. The periodicity conditions become

$$e^{ip_1 N} = A(p_2,p_1), \qquad\qquad e^{ip_2 N} = A(p_1,p_2). \tag{26}$$

These are called the Bethe equations and they are simply the equivalent of the quantization condition of particles on a sphere. Their interpretation is rather clear. You take one of the particles and you move it around the circle. Once you encounter the other particle, you scatter with it and pick up the scattering phase. Then, when you've moved around your wave function is unchanged up to a phase.

Moreover, notice that by multiplying the two equations you derive that $e^{i(p_1+p_2)L} = 1$. Because of translation invariance this is expected. Finally, let us introduce the so-called rapidity $u = 2\cot\frac{p}{2}$, then the S-matrix becomes rational

$$A = \frac{u_1 - u_2 - i}{u_1 - u_2 + i} \tag{27}$$

and the Bethe equations simply become a set of polynomial equations

$$\left[\frac{u_1 + \frac{i}{2}}{u_1 - \frac{i}{2}}\right]^N = \frac{u_1 - u_2 + i}{u_1 - u_2 - i}, \qquad\qquad \left[\frac{u_2 + \frac{i}{2}}{u_2 - \frac{i}{2}}\right]^N = \frac{u_2 - u_1 + i}{u_2 - u_1 - i}. \tag{28}$$

To now find the spectrum of the Hamiltonian you have to solve the Bethe equations for the momentum and then simply substitute it into the energy. So, let us find the exact two-magnon spectrum. There are three types of solutions

- Descendants, where $p_1 = 0$
- Bound States, where $p_{1,2}$ are complex conjugate
- Two magnon states

Let us discuss each kind of solution in detail.

First, there are the descendants. These are simply part of the $\mathfrak{su}(2)$ muliplet that contain the single magnon states. There are obtained by acting with $S^-$ on them and as a consequence, there are $N$ of them. Their energy is of course the same as the corresponding single magnon one.

Taking a logarithm of the Bethe equations gives us

$$Np_1 = 2\pi\lambda_1 + \theta, \qquad\qquad Np_1 = 2\pi\lambda_1 - \theta, \tag{29}$$

where $\theta = -i \log A$ satisfies

$$2\cot\frac{\theta}{2} = \cot\frac{p_1}{2} - \cot\frac{p_2}{2}. \tag{30}$$

We have to find all integers $\lambda_1, \lambda_2$ such that the above equations are satisfied. Clearly we can restrict to

$$0 < \lambda_1 \leq \lambda_2 < L, \tag{31}$$

and we are left with a finite number of integer to check. A more detailed look at these equations reveal that there are $L - 3$ bound states. Solutions with complex conjugate momenta are interpreted as bound states since

$$p_1 = \tfrac{1}{2}p + iv, \qquad\qquad p_1 = \tfrac{1}{2}p - iv, \tag{32}$$

thus we see that both particles move with the same momentum. Furthermore, because of the imaginary part the wave function falls of when the distance between the two magnons increases.

**Multi-magnons.** The fact that this spin chain in integrable can be seen when adding more magnons. Namely, it turns out that just knowing the dispersion relation (energy) and the two-magnon scattering matrix $\theta$ completely fixes the spectrum.

It turns out that the eigenvalues of the Hamiltonian take the form

$$|p_1, \ldots, p_M\rangle = \sum_{n_1 < n_2 < \ldots} a(n_1, \ldots, n_M) S^-_{n_1} \ldots S^-_{n_M} |0\rangle. \tag{33}$$

with the specific choice

$$a(n_1, \ldots, n_M) = \sum_{\sigma \in S_M} e^{ip_{\sigma_i} n_i + i \sum_{i<j} \theta_{\sigma_i \sigma_j}}. \tag{34}$$

This remarkable feature can be seen as a defining feature of integrable models. All the relevant data of the scattering and spectrum is fully encoded in the dispersion relation and the two-body scattering.

**Bethe equations.** Now that we know the full eigenfunction, let us write down the corresponding Bethe equations that arise from the periodicity conditions

$$e^{ip_i N} = \prod_{j \neq i} A(p_j, p_i). \tag{35}$$

Again the interpretation is rather simple and corresponds to moving the $i$-th particle around the spin chain. Written out in terms of the rapidity it simply become

$$\left[\frac{u_i + \frac{i}{2}}{u_i - \frac{i}{2}}\right]^N = \prod_{i \neq j} \frac{u_i - u_j + i}{u_i - u_j - i}. \tag{36}$$

The energy in terms of the rapidity is given by

$$E = \frac{2J}{4 + u^2}. \tag{37}$$

The spectrum can then be found by solving the Bethe equations (which is simply a set of algebraic equations) and summing the energies of the different magnons.

Summarizing, we found a way of exactly encoding the spectrum of the full XXX spin chain purely in a set of algebraic relations. Even more useful is the fact that we can take limits in this way, for example the thermodynamic limit $N, M \to \infty$ which is relevant for practical applications.

# 4 Monodromy and R-matrix

In this section we will use a different approach to study the Heisenberg spin chain. With this approach the integrability of the model becomes more transparent. Furthermore, the results and constructions are easier to generalize in this formalism.

**Lax operator.** The basic tool of the algebraic Bethe Ansatz approach is the so-called Lax operator $L$. Consider again a chain with $N$ sites and corresponding Hilbert space $\mathcal{H} = \bigotimes_i V_i$. These spaces we call quantum spaces, which for us will be $\mathbb{C}^2$. To this chain we add an additional auxiliary space $V_a$, which in our case will again simply be $\mathbb{C}^2$. The Lax operator is an operator $L : V_i \otimes V_a \to V_i \otimes V_a$ and for our model it is of the form

$$L_{n,a}(u) = u\,1 + iS^a_n \otimes \sigma^a, \tag{38}$$

where $\sigma^\alpha$ are the Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{39}$$

For spin-$\frac{1}{2}$ they are related to the spin operators as $S^a = \frac{1}{2}\sigma^a$. Now remember from classical mechanics that the Lax matrix was a matrix with entries that were functions on the phase space. Here we see something similar, we can view the Lax operator $L$ as a matrix on the auxiliary space with entries

$$L_{n,a} = \begin{pmatrix} u + iS^z_n & iS^-_n \\ iS^+_n & u - iS^z_n \end{pmatrix}. \tag{40}$$

This has exactly the same structure, the entries are now operators acting on the physical space. The interpretation of the auxiliary space is simply the space from which the matrix structure of $L$ originates.

The next ingredient we need is the commutation relation between the entries of the Lax operator. This is the equivalent of the Poisson bracket between the entries of the Lax matrix in classical mechanics. Let us introduce the permutation operator

$$P = \tfrac{1}{2}(1 + \sigma^a \otimes \sigma^a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{41}$$

then we can rewrite

$$L_{n,a} = (u - \tfrac{i}{2})1 + iP_{n,a}. \tag{42}$$

The operator $P$ is called the permutation operator because

$$P(x \otimes y) = y \otimes x. \tag{43}$$

Now, since we know how the spin operators commute, namely

$$[S^a, S^b] = i\epsilon^{abc} S^c, \tag{44}$$

we can explicitly work out the commutation relations of the Lax matrix. They can be written in the following compact form

$$R_{ab}(u_1 - u_2) L_{n,a}(u_1) L_{n,b}(u_2) = L_{n,b}(u_2) L_{n,a}(u_1) R_{ab}(u_1 - u_2), \tag{45}$$

where the operator $R$ is called the quantum R-matrix and it has the following form

$$R_{ab} = \lambda 1 + iP_{ab}. \tag{46}$$

This is exactly the type of relation that we also saw for the Sine-Gordon model. Relation (45) is called the fundamental commutation relation. This is most easily checked by using the formulation in terms of permuation operators

$$\begin{aligned} &((u_1 - u_2)1 + iP_{ab})((u_1 - \tfrac{i}{2})1 + iP_{n,a})((u_2 - \tfrac{i}{2})1 + iP_{n,b}) = \\ &\quad ((u_2 - \tfrac{i}{2})1 + iP_{n,b})((u_1 - \tfrac{i}{2})1 + iP_{n,a})((u_1 - u_2)1 + iP_{ab}), \end{aligned} \tag{47}$$

and using $P_{n,a_1} P_{n,a_2} = P_{a_1,a_2} P_{n,a_1} = P_{n,a_2} P_{a_2,a_1}$ and $P_{a,b} = P_{b,a}$.

From the fundamental commutation relation you can show that the R-matrix needs to satisfy the quantum Yang-Baxter equation. Consider the following product of $L$-operators (we drop the index $n$ of the physical space for the moment)

$$L_1 L_2 L_3 = R_{12}^{-1} L_2 L_1 L_3 R_{12} = R_{12}^{-1} R_{13}^{-1} L_2 L_3 L_1 R_{13} R_{12} = R_{12}^{-1} R_{13}^{-1} R_{23}^{-1} L_3 L_2 L_1 R_{23} R_{13} R_{12}, \tag{48}$$

while on the other hand

$$L_1 L_2 L_3 = R_{23}^{-1} L_1 L_3 L_2 R_{23} = R_{23}^{-1} R_{13}^{-1} L_3 L_1 L_2 R_{13} R_{23} = R_{23}^{-1} R_{13}^{-1} R_{12}^{-1} L_3 L_2 L_1 R_{12} R_{13} R_{23}, \tag{49}$$

Hence, for these both relations to coincide we impose that our R-matrix satisfies the equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \tag{50}$$

In the semi-classical limit this reduces to the classical Yang-Baxter equation.

**Monodromy and transfer matrix.** Our spin chain has $N$ sites. Then, using our Lax operator let us define the monodromy matrix

$$T_a = L_{N,a}(u) \dots L_{1,a}(u). \tag{51}$$

It can be seen as a two by two matrix acting on the auxiliary space whose entries are operators that act on the physical Hilbert space. We write it as

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{52}$$

From this monodromy we can derive the tower of conseved charges that characterize intergable systems.

First of all, we need the commutation relations between the different entries of the monodromy matrix. This can be found by making repeated use of the FCR for the Lax operator. We find

$$R_{ab}(u_1 - u_2)T_a(u_1)T_b(u_2) = T_b(u_2)T_a(u_1)R_{ab}(u_1 - u_2). \tag{53}$$

Let us introduce the transfer matrix

$$t = \mathrm{tr}_a T_a = A + D. \tag{54}$$

Rewriting the FCR relations in the following way

$$T_a(u_1)T_b(u_2) = R_{ab}(u_1 - u_2)^{-1}T_b(u_2)T_a(u_1)R_{ab}(u_1 - u_2), \tag{55}$$

and subsequently taking the trace over both auxiliary spaces we find by cyclicity of the trace that

$$[t(u_1), t(u_2)] = 0. \tag{56}$$

Since the transfer matrix depends on a spectral parameter, we can simply make a power series in $u_1, u_2$, resulting in a family of commuting operators. Of course, from our usual definition of integrable systems, we would like the Hamiltonian to be one of them.

In order to see that this is the case, we expand the transfer matrix around the special point $u = \frac{i}{2}$. To first order we find

$$T_a(\tfrac{i}{2}) = i^N P_{N,a}P_{N-1,a} \dots P_{1,a} = i^N P_{12}P_{23} \dots P_{N-1,N}P_{N,a}, \tag{57}$$

where we used the various properties of the permutation operator listed below (47). Now it is easy to take the trace of the auxiliary space and since $\mathrm{tr}_a P_{N,a} = 1_N$. Thus we find

$$t(\tfrac{i}{2}) = i^N P_{12}P_{23} \dots P_{N-1,N} = \mathcal{U}, \tag{58}$$

which simply is the shift operator. Actually, by definition the shift operator is identified with the momentum operator as

$$\mathcal{U} = e^{ip}. \tag{59}$$

Next, we look at the next order in our expansion and we take the derivative of the transfer matrix at the point $\frac{i}{2}$. Direct computation gives

$$\left.\frac{dT_a}{du}\right|_{\lambda=\frac{i}{2}} = i^{N-1}\sum_n P_{N,a} \dots \hat{P}_{n,a} \dots P_{1,a} = i^{N-1}\sum_n P_{12}P_{23} \dots P_{n-1,n+1} \dots P_{N-1,L}P_{N,a}, \tag{60}$$

where $\hat{P}$ means that the corresponding term is absent. The using the expression for $t(\frac{i}{2})$ we quickly find

$$\frac{d}{du}\ln t(u)\Big|_{u=\frac{i}{2}} = -i\sum_n P_{n,n+1}. \tag{61}$$

Now, since we can express $P$ in terms of Pauli-matrices it is not difficult to rewrite the Hamiltonian in term of permutation operators as follows

$$H = -J\left[\tfrac{1}{2}\sum_n P_{n,n+1}\right]. \tag{62}$$

We immediately recognize the logarithmic derivative and hence the transfer matrix generates a tower of conserved commuting quantities and we are dealing with an integrable system.

# 5 Algebraic Bethe Ansatz

What remains is finding the spectrum of the Hamiltonian. In this formalism however, we can do slightly better. Since the tranfer matrix defines a set of commuting quantities, we can diagonalize them simultaneously. In particular, we will find the eigenvalues and eigenstates of the transfer matrix directly. Since the Hamiltonian can be constructed from $t$, we automatically find the spectrum of the Hamiltonian, but not only that, we actually find the spectrum of all the conserved commuting charges.

The spectrum is found by using a variation of the Bethe Ansatz, called the algebraic Bethe Ansatz. It uses different elements from the monodromy matrix to make an Ansatz for the eigenstates that depend on some rapidities (momenta) which need to satisfy the Bethe equations.

**Ground state.** Consider again the state where all spins are flipped up. It is an eigenstate of the transfer matrix. In particular, from the explicit expression of our Lax operator we readily derive

$$A|0\rangle = \prod_n (u+iS_n^z)|0\rangle = (u+\tfrac{i}{2})^N|0\rangle, \qquad D|0\rangle = (u-\tfrac{i}{2})^N|0\rangle, \qquad C|0\rangle = 0. \tag{63}$$

Hence, $|0\rangle$ is an eigenstate with eigenvalue

$$t|0\rangle = (A+D)|0\rangle = \left[(u+\tfrac{i}{2})^N + (u-\tfrac{i}{2})^N\right]|0\rangle. \tag{64}$$

This is very similar to the coordinate Bethe Ansatz. However, now we need to find a way to flip spins. We will do this by using the operator $B$.

**Commutation relations.** The key ingredient of the algebraic Bethe Ansatz are the commutation relations between the different operators from the monodromy matrix. Let us work the relevant ones out explicitly. We have

$$T_1(u) = \begin{pmatrix} A(u) & 0 & B(u) & 0 \\ 0 & A(u) & 0 & B(u) \\ C(u) & 0 & D(u) & 0 \\ 0 & C(u) & 0 & D(u) \end{pmatrix}, \qquad T_2(v) = \begin{pmatrix} A(v) & B(v) & 0 & 0 \\ C(v) & D(v) & 0 & 0 \\ 0 & 0 & A(v) & B(v) \\ 0 & 0 & C(v) & D(v) \end{pmatrix}. \tag{65}$$

Then, by the explicit form of the R-matrix

$$R = \begin{pmatrix} u-v+i & 0 & 0 & 0 \\ 0 & u-v & i & 0 \\ 0 & i & u-v & 0 \\ 0 & 0 & 0 & u-v+i \end{pmatrix}, \tag{66}$$

the fundamental commutation relation simply the following relations between the operators $A, B, D$

$$B(u)B(v) = B(v)B(u), \tag{67}$$

$$A(u)B(v) = \frac{u-v-i}{u-v}B(v)A(u) + \frac{i}{u-v}B(u)A(v), \tag{68}$$

$$D(u)B(v) = \frac{u-v+i}{u-v}B(v)D(u) - \frac{i}{u-v}B(u)D(v). \tag{69}$$

From the form of the Lax operator we see that we can interpret $B$ as some kind of creation operator that creates magnons. In view of this, we make the Ansatz

$$|u_1, \ldots, u_M\rangle = B(u_1)\ldots B(u_M)|0\rangle. \tag{70}$$

Now we have to compute the action of the transfer matrix on this and investigate when this will be an eigenvector of $t$ (and hence of the Hamiltonian). The tranfer matrix is simply given by $A+D$ and we already know how they act on the vacuum. And by making use of the commutation relations it is then easy to find

$$A(u)|u_1, \ldots, u_M\rangle = (u+\tfrac{i}{2})^N \prod_n \frac{u-u_n-i}{u-u_n}|u_1, \ldots, u_M\rangle + \sum_n W_n^A(u,u_i)|\ldots u_{n-1}, u, u_{n+1}\ldots\rangle \tag{71}$$

for some coefficient $W_n^A$. Since the operators $B$ commute with each other we can write

$$|u_1, \ldots, u_M\rangle = B(u_n)\prod_{i\neq n} B(u_i)|0\rangle, \tag{72}$$

for any $n$. Thus the only term that can contribute to $W_n^A$ is the one where $A$ and $B$ commute but interchange spectral parameters and then the $A$ operator simply commutes through all the other $B$'s. By this argument we find

$$W_n^A = \frac{i(u_n+\frac{i}{2})^N}{u-u_n}\prod_{j\neq n}\frac{u_n-u_j-i}{u_n-u_j} \tag{73}$$

The discussion with the commutators involving the $D$-operators is completely analogous and we get

$$D(u)|u_1, \ldots, u_M\rangle = (u-\tfrac{i}{2})^N \prod_n \frac{u-u_n+i}{u-u_n}|u_1, \ldots, u_M\rangle + \sum_n W_n^A(u,u_i)|\ldots u_{n-1}, u, u_{n+1}\ldots\rangle \tag{74}$$

where

$$W_n^D = -\frac{i(u_n-\frac{i}{2})^N}{u-u_n}\prod_{j\neq n}\frac{u_n-u_j+i}{u_n-u_j}. \tag{75}$$

Combining these two result, we see that the state $|u_1, \ldots, u_M\rangle$ is indeed an eigenstate with eigenvalue $\Lambda$

$$\Lambda = (u + \tfrac{i}{2})^N \prod_j \frac{u - u_j - i}{u - u_j} + (u - \tfrac{i}{2})^N \prod_j \frac{u - u_j + i}{u - u_j} \tag{76}$$

Provided that the term $W_n^A + W_n^D$ vanishes. This term is usually referred to as the 'unwanted' term. The requirement that this vanishes puts restrictions on the parameters $u_i$ in the form of a set of equations

$$\left[\frac{u + \frac{i}{2}}{u - \frac{i}{2}}\right]^N = \prod_{j \neq n} \frac{u_n - u_j + i}{u_n - u_j - i}, \tag{77}$$

which are exactly the Bethe equations. We see that $B$ can be viewed as a magnon creation operator.

Finally, notice that the Bethe equations can also be read off from the eigenvalue itself. If we want our eigenvalue to be an analytic function of $u$, then is should not have any poles. However, at $u = u_j$ it clearly has an apparent pole. However, making sure this pole is only superficial we need to impose the Bethe equations. In other words, the Bethe equations can also be found from the analytical properties of the transfer matrix.

**Dispersion relation.** The last ingredient we need is to find the spectrum. We have an explicit expression for the eigenvalues, so let us determine both the energy and momentum. First, recall that the momentum was defined via

$$e^{ip} = t(\tfrac{i}{2}) \Rightarrow p = -i \ln t(\tfrac{i}{2}). \tag{78}$$

Substituting the eigenvalue into this equation then results in

$$p = \sum_j p_j, \qquad p_j = i \log \frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}} \quad \Leftrightarrow \quad u = 2 \cot \frac{p}{2}. \tag{79}$$

This is exactly the relation between momentum and rapidity that we encountered in the coordinate Bethe Ansatz. The energy is found from the logaritmic derivative and gives

$$E = \sum_{n=1}^{M} \frac{1}{4u^2 + 1}, \tag{80}$$

which, in turn, perfectly agrees with the energy we found in the coordinate Bethe Ansatz. We already found that the Bethe equations agree and thus we have successfully reproduced the spectrum of the Hamiltonian that we found with the help of the coordinate Bethe Ansatz.

# 6 Relation between R and L

The crucial ingredient of the algebraic Bethe Ansatz method were the fundamental commutation relations, which are fully described just in terms of the R-matrix. You can then ask whether any R-matrix that satisfies the Yang-Baxter equation will define an integrable spin chain. The

answer to this is positive. To see this, suppose that we have a R-matrix $R_{ij}(u_i - u_j)$ that satisfies the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{81}$$

and at some particular point $\lambda$ reduces to the permutation operator

$$R(\lambda) = P. \tag{82}$$

This defines an integrable spin chain with a nearest neighbour Hamiltonian given by

$$\mathcal{H} = \sum_n \mathcal{H}_{n,n+1}, \qquad \mathcal{H}_{n,n+1} = R^{-1}_{n,n+1}(u)\frac{d}{du}R_{n,n+1}\Big|_{u=\lambda}. \tag{83}$$

This can be shown as follows. Define the Lax operator $L$ to be

$$L_{n,a}(u) = R_{a,n}(u - \mu), \tag{84}$$

where $\mu$ is some constant. Then, by the Yang-Baxter equation, $L$ satisfies fundamental commutation relations of the form

$$R_{ab}(u_1 - u_2)L_{n,a}(u_1)L_{n,b}(u_2) = L_{n,b}(u_2)L_{n,a}(u_1)R_{ab}(u_1 - u_2) \tag{85}$$

Next we define the monodromy and transfer matrix in the usual way

$$T_a = L_{N,a}\dots L_{1,a}, \qquad t = \mathrm{tr}_a T_a. \tag{86}$$

Then, by the fundamental commutation relations the transfer matrix defines a family of commuting quantities. Since $R(\lambda)$ is the special point where the R-matrix trivialises, the transfer matrix at the point $\nu = \lambda - \mu$ simply becomes the shift operator

$$t(\nu) = \mathcal{U} = e^{iP}. \tag{87}$$

Next let us define the Hamiltonian as the logarithmic derivative at $\nu$. We first consider the derivative of the monodromy matrix, which is readily found to be

$$\frac{dT_a}{du}(\nu) = \sum_n P_{N,a}\dots\frac{dL_{n,a}}{du}(\nu)\dots P_{1,a} = \sum_n \frac{dL_{n,n+1}}{du}(\nu)P_{1,2}\dots\dots P_{N-1,N}P_{N,a} \tag{88}$$

Since $\mathrm{tr}_a P_{N,a} = 1$ and $P^2_{ab} = 1$ we quickly find

$$\mathcal{H} \equiv \frac{dt}{du}(\nu)t^{-1}(\nu) = \sum_n \frac{dL_{n,n+1}}{du}(\nu)P_{n,n+1} \tag{89}$$

Recalling the relation between the Lax operator and the R-matrix and by swtiching the permution with the Lax operator we then arrive at the desired result. Moreover, chosing a different normalization of the R-matrix simply amounts to adding a constant to the Hamiltonian, which does not affect the physics of the system.

# 7 Symmetries from T

## 7.1 The $\mathfrak{su}(2)$ algebra

The above construction revolved about the point that at $\frac{i}{2}$, the Lax operator (and consequently the monodromy matrix) become rather simple. There is a second point where this happens, namely at $u \to \infty$. In fact at this point $L \to \lambda 1$ and expanding the monodromy matrix leads to the following

$$T(u) \xrightarrow{u\to\infty} u^N + u^{N-1} i \sum_n S_n^a \otimes \sigma^a + \dots \tag{90}$$

Recall from our definition of the coordinate Bethe Ansatz that we can realize the total spin operators (that form an $\mathfrak{su}(2)$ algebra) were defined exactly as $S^a = \sum_n S_n^a$ and hence

$$T \to u^N + u^{N-1} i \sum_n S^a \otimes \sigma^a + \dots . \tag{91}$$

In other words, we see some symmetry generator appearing in this limit. Remember that last week we explicitly checked that $[\mathcal{H}, S^a] = 0$ the total spin operators commute with the Hamiltonian and as such describe some of its symmetries.

Let us make this more explicit. Let us consider the FCR at the limit where we send one of the rapidities to $\infty$. We get

$$[(u-v) + iP_{ab}]\, T_a(u)(v^N + iS^a \otimes \sigma^a)_b = (v^N + iS^a \otimes \sigma^a)_b T_a(u)\, [(u-v) + iP_{ab}] \tag{92}$$

The leading order terms cancel out and we are left with the subleading terms. They give (we express $P$ via the Pauli matrices)

$$[T(u), S^a + \tfrac{1}{2}\sigma^a] \otimes \sigma^a = 0. \tag{93}$$

This describes how the different terms of the monodromy matrix transform under the global symmetry generators

$$[S^a, T] = \tfrac{1}{2}[T, \sigma^a]. \tag{94}$$

Thus, for example we find

$$[S^z, T] = \tfrac{1}{2}[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}] = \begin{pmatrix} 0 & -B \\ C & 0 \end{pmatrix}. \tag{95}$$

Working out these relations then leaves us with the following commutation relations

$$[S^z, B] = -B, \qquad [S^+, B] = A - D. \tag{96}$$

We can easily check from the fundamental commutation relations that the transfer matrix (and thus the Hamiltonian as well) displays $\mathfrak{su}(2)$ symmetry. We have that

$$[S^z, A] = [S^z, D] = 0, \qquad [S^+, A] = -[S^+, D] = -C, \qquad [S^-, A] = -[S^-, D] = B, \tag{97}$$

from which it easily follows that

$$[S^a, t] = [S^a, A + D] = 0. \tag{98}$$

This type of symmetry is called local since the terms of the total spin operators only act on one site.

## 7.2 Highest weight states

Not only can we check that the Hamiltonian respects $\mathfrak{su}(2)$ symmetry, but since we also know how the spin operators commute with the creation operator $B$. By these commutation relations we can study the action of the symmetry operators on the eigenstates of the transfer matrix. In particular we will show that they are highest weight states. In other words, the eigenstates created in this way are not descendants.

First, it is easy to see that the ground state $|0\rangle$ is a heighest weight state since

$$S^+|0\rangle = 0, \qquad S^z|0\rangle = \frac{N}{2}|0\rangle. \tag{99}$$

The general state $|u_1, \dots u_M\rangle$, is similarly an eigenstate of $S^z$ and its eigenvalue readily follows from the commutator of $B$ with $S^z$

$$S^z|u_1, \dots u_M\rangle = (\frac{N}{2} - M)|u_1, \dots u_M\rangle. \tag{100}$$

Next we need to show that $S^+|u_1, \dots u_M\rangle = 0$. We can write

$$\begin{aligned} S^+|u_1, \dots u_M\rangle &= \sum_j B(u_1) \dots (A(u_j) - D(u_j)) \dots B(u_M)|0\rangle \\ &= \sum_j O_j B(u_1) \dots \hat{B}(u_j) \dots B(u_M)|0\rangle \end{aligned} \tag{101}$$

Then by similar arguments as in the case for the Algebraic Bethe Ansatz one can show that $O_j = 0$ if one imposes the Bethe equations.

Summarizing, because of the $\mathfrak{su}(2)$ symmetry, the eigenstates of the Hamiltonian split into irreducible representations of this algebra. The Bethe vectors constructed with the help of the Algebraic Bethe Ansatz provide heighest weight states and the rest of the representation consists of descendents.

## 7.3 Non-local symmetries

We derived the algebra simply by looking at the leading contribution at $u \to \infty$. Of course, we can go beyond this and study the next order. One quickly finds

$$T(u) \xrightarrow{u\to\infty} u^N + u^{N-1} i \sum_n S_n^a \otimes \sigma^a - u^{N-2} \sum_{m<n} S_m^a S_n^b \otimes \sigma^a \sigma^b. \tag{102}$$

We see the appearance of a non-local term. From the explicit form of the Pauli-matrices we see that there are only three independent symmetries of this form

$$\sum_{m<n} (S_m^+ S_n^- - S_m^- S_n^+) \otimes \sigma^z, \quad \sum_{m<n} (S_m^z S_n^- - S_m^- S_n^z) \otimes \sigma^-, \quad \sum_{m<n} (S_m^z S_n^+ - S_m^+ S_n^z) \otimes \sigma^+. \tag{103}$$

These are in a sense the next level equivalents of $S^{z,-,+}$. Let us denote them by $\hat{S}^a$. Now before studying the various commutation relations, let us see whether these non-local charges are symmetries of the Hamiltonian.

Let us check this for $\hat{S}^z$. We find

$$[\mathcal{H}, \hat{S}^z] = \sum_{i<j} \sum_n [S_n^a S_{n+1}^a, S_i^+ S_j^- - S_i^- S_j^+] \tag{104}$$

When $|i-j| \geq 2$ the above commutator vanishes since it reduces to the commutator

$$[S_n^a S_{n+1}^a, S_i^{\pm}], \tag{105}$$

which is zero since $S^{\pm}$ are symmetries of the Hamiltonian by themselves. Hence, the only non-trivial remaining term is

$$[\mathcal{H}, \hat{S}^z] = \sum_n [S_n^a S_{n+1}^a, S_n^+ S_{n+1}^- - S_n^- S_{n+1}^+] = \sum_n S_n^z - S_{n+1}^z. \tag{106}$$

Thus, whether $\hat{S}^z$ is a symmetry actually depends on the boundary conditions on the spin chain. This operator clearly vanishes on periodic chains or on infinitely large chains and consquently, we have found a non-trivial non-local symmetry of the Hamiltonian.

### 7.4 Commutation relations

Now that we have established a whole new set of symmetries, let us look at what kind of algebra they generate. The commutation relations of the $\mathfrak{su}(2)$ algebra are simply

$$[S^z, S^+] = S^+, \qquad [S^z, S^-] = -S^-, \qquad [S^+, S^-] = 2S^z. \tag{107}$$

Let us work out the commutator $[\hat{S}^z, S^+]$

$$\begin{aligned}
[\hat{S}^z, S^+] &= \sum_{k,m<n} [S_m^+ S_n^- - S_m^- S_n^+, S_k^+] \\
&= \sum_{k,m<n} \delta_{k,n} S_m^+ [S_n^-, S_k^+] + \delta_{k,m} [S_m^-, S_k^+] S_n^+ \\
&= \sum_{m<n} S_m^z S_n^+ - S_m^+ S_n^z \\
&= \hat{S}^+.
\end{aligned} \tag{108}$$

This looks exactly like the commutator of the normal algebra. In fact, we can work out all commutation relations to find

$$[\hat{S}^z, S^+] = [S^z, \hat{S}^+] = \hat{S}^+, \quad [\hat{S}^z, S^-] = [S^z, \hat{S}^-] = -\hat{S}^-, \quad [\hat{S}^+, S^-] = [S^+, \hat{S}^-] = 2\hat{S}^z. \tag{109}$$

This looks like a standard loop algebra. However, when commuting two higher level symmetries we depart from this simple structure. For example

$$[\hat{S}^+, \hat{S}^z] = \hat{\hat{S}}^+ + \text{cubic terms in } S, \tag{110}$$

where $\hat{\hat{S}}^+$ would be the operator at a higher level again. The algebra that has a structure of this form is a so-called Yangian algebra, which is a Hopf algebra.

## 8 Yangians and Hopf algebras

Symmetries in physical systems are generically described by so-called Lie algebras. This means that one can define an action of elements of the Lie algebras on physical states, *i.e.* they transform in a so-called representation of the algebra.

**Lie algebras.** A Lie algebra $\mathfrak{g}$ is a vector space, together with a Lie bracket $[,]$. Let $J^A$ be a basis of $\mathfrak{g}$. The structure constants $f^{AB}{}_C$ are defined as

$$[J^A, J^B] = f^{AB}{}_C J^C \tag{111}$$

Moreover, the Lie bracket needs to satisfy the Jacobi identity

$$[J^A, [J^B, J^C]] + [J^B, [J^C, J^A]] + [J^C, [J^A, J^B]] = 0. \tag{112}$$

Now, of course we already saw all these properties all the way back when we were dealing with Poisson brackets in classical mechanics.

A representation of a Lie algebra is a vector space $V$ together with a map of the Lie algebra into the automorphisms of that vector space.

**Associative unital algebras.** A more general notion of that of an associative (unital) algebra. Such an algebra $A$ is a vector space together with a multiplication $m : A \otimes A \to A$

$$m : x \otimes y \mapsto xy. \tag{113}$$

The algebra is called associative is the multiplication is associative, *i.e.*

$$x(yz) = (xy)z. \tag{114}$$

and is furthermore unital if it admits a unit 1

$$x = 1x = x1. \tag{115}$$

To any associative algebra you can assign a corresponding Lie algebra. Consder the so-called universal enveloping algebra $U(A)$, which contains the algebra and all products of elements. Then $U(A)$ can be equipped with a Lie bracket as follows

$$[a, b] \equiv ab - ba. \tag{116}$$

The Jacobi identity follows from associativity.

**Hopf algebra.** A Hopf algebra is an algebra that not only has a multiplication but also a comultiplication $\Delta : A \to A \otimes A$[1].

The coproduct tells you how to take an algebra generator and how to map it into the tensor product. This describes how the algebra acts on multi-particle states. The coproduct needs to be co-associative

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta \tag{117}$$

and be compatible with the multiplicative structure of the algebra

$$\Delta(ab) = \Delta(a)\Delta(b). \tag{118}$$

Any Lie algebra can be given the structure of a Hopf algebra by introducing the following coproduct

$$\Delta J = J \otimes 1 + 1 \otimes J. \tag{119}$$

However, if there are symmetries of non-local form, the coproduct becomes much more involved and leads to non-trivial representations.

[1] A Hopf algebra furthermore has a counit and antipode, but we will we not specify their properties here.

**Yangian.** Consider a Lie algebra $\mathfrak{g}$ with generators $J^A$ and structure constants $f^{AB}{}_C$. The Yangian is then defined by adding a second set of generators $\hat{J}^A$ that satisfy the following relations

$$[J^A, \hat{J}^B] = f^{AB}{}_C \hat{J}^C. \tag{120}$$

Furthermore, the equivalent of the Jacobi identity when some of the operators are hatted is modified

$$[J^A, [J^B, J^C]] + [J^B, [J^C, J^A]] + [J^C, [J^A, J^B]] = \mathcal{O}(J^3), \tag{121}$$

where the right hand side is some known cubic combination of Lie algebra generators, which I won't spell out[2]. We can give it the following coproduct

$$\Delta \hat{J}^A = \hat{J}^A \otimes 1 + 1 \otimes \hat{J}^A + f^A{}_{BC} J^B \otimes J^C. \tag{122}$$

We recognize the form of the non-local symmetries of the XXX spin chain. The Yangian is an infinite dimensional algebra. Namely the commutator between two Yangian generators is again a new generator and so on.

**Evaluation representation.** A very useful representation of a Yangian is the so-called evaluation representation. Consider a representation $\rho$ of the Lie algebra $\mathfrak{g}$, then we set

$$\rho(\hat{J}^A) = u\rho(J^A), \tag{123}$$

where $u$ is a spectral parameter. This identification does not necessarily work for all represenations $\rho$.

# 9 R-matrix from Yangian symmetry

Let us now discuss how we can use the Yangian algebra to derive the R-matrix. First let us make a link between symmetries of the Hamiltonian and the R-matrix. Suppose the $R$-matrix has a symmetry of the form

$$RQ = PQPR, \tag{124}$$

then by the relation of the Hamiltonian and the R-matrix, $Q$ is a symmetry of the Hamiltonian

$$[H, Q] = 0. \tag{125}$$

We already saw that the Hamiltonian exhibits Yangian symmetry, so let us now try to use it to find the R-matrix.

**The algebra** First of all, we should require $\mathfrak{su}(2)$ symmetry. These generators have a standard coproduct, which is left invariant under permutation. It results in the following equations for the R-matrix

$$[R, S^a \otimes 1 + 1 \otimes S^a] = 0. \tag{126}$$

You can quickly check that this means that the R-matrix is of the following form

$$R = a1 + P. \tag{127}$$

We have used the freedom of choosing our normalization by setting the coefficient in front of the permuation equal to unity. There is only the unknown coefficient $a$ left to determine.

[2]See for example http://arxiv.org/abs/hep-th/0409183 for the exact form.

**The Yangian** The coproduct of the Yangian however, is not invariant under permutations. Our two auxiliary spaces have two different spectral parameters $u_1, u_2$ and we implement them in the Yangian by using the evaluation representation. Then we get

$$\begin{aligned}\Delta\hat{S}^z &= u_1 S^z \otimes 1 + 1 \otimes u_2 S^z + i(S^+ \otimes S^- - S^- \otimes S^+),\\ P(\Delta\hat{S}^z)P &= u_1 S^z \otimes 1 + 1 \otimes u_2 S^z - i(S^+ \otimes S^- - S^- \otimes S^+).\end{aligned} \tag{128}$$

We need to compute

$$\Delta\hat{S}^z R = P(\Delta\hat{S}^z)PR. \tag{129}$$

Remarkably we find that Yangian invariance fixes the coefficient $a$ uniquely and we get

$$R = (u_1 - u_2)1 + iP. \tag{130}$$

We see that the R-matrix and consequently the entire model is fixed simply by looking at the symmetries.