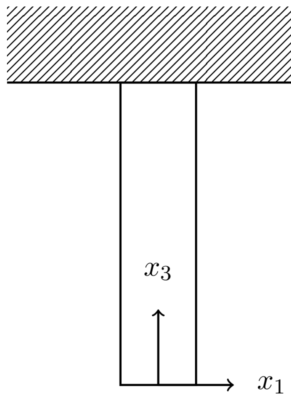
Exercise 1. *Deformation of an elastic bar in homogeneous gravitational field*

Determine the displacement field $u = (u_1, u_2, u_3)$ of a homogeneous bar of length L with arbitrary cross section. The bar is hanging from the top under the influence of gravity $F = -\rho g(0, 0, 1)$. The bottom and lateral surfaces are free. There is no boundary condition for u specified on the top. While this being not quite realistic, it allows to solve the problem using a simple ansatz for the stress $\sigma_{ik}(x)$. This can be done in the following steps:



(i) Make an ansatz for $\sigma_{ik}(x)$ similarly to the 'axial pull to a solid cylinder' from the lecture and deduce from the equilibrium condition that

$$\varepsilon_{11} = \varepsilon_{22} = -\frac{\nu\rho g}{E} x_3, \qquad \varepsilon_{33} = \frac{\rho g}{E} x_3, \tag{1}$$

where ν and E are the Poisson coefficient and the Young's modulus respectively.

Hint: Use the relation between stress and strain from the lecture given by

$$\varepsilon_{ik} = \frac{1+\nu}{E}\sigma_{ik} - \frac{\nu}{E}\mathrm{tr}(\sigma)\delta_{ik}. \tag{2}$$

(ii) Use the relation

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3}$$

to find the following expressions for u_1, u_3

$$u_1 = -\frac{\nu\rho g}{E} x_1 x_3 + f_1(x_2, x_3), \qquad u_3 = \frac{\rho g}{2E} x_3^2 + f_3(x_1, x_2), \tag{4}$$

for some functions f_1, f_3 and show that f_3 satisfies both of

$$f_3(x_1, x_2) = \frac{\nu\rho g}{2E} x_1^2 + g_1(x_2)x_1 + h_1(x_2), \qquad f_3(x_1, x_2) = \frac{\nu\rho g}{2E} x_2^2 + g_2(x_1)x_2 + h_2(x_1), \tag{5}$$

for some functions g_1, g_2, h_1, h_2.

(iii) Find a solution u such that $u(0, 0, L) = 0$.

(iv) In case of the material being Aluminium ($\rho = 2.7\,\mathrm{g/cm^3}$, Yield strength $= 400\,\mathrm{MPa}$), determine the maximal length L the bar can have without breaking.

Exercise 2. ***Anisotropic homogeneous media***

For the present exercise it is of advantage to introduce the following notation.

$$\varepsilon_i := \varepsilon_{ii}, \qquad \varepsilon_4 := 2\varepsilon_{23}, \quad \varepsilon_5 := 2\varepsilon_{31}, \quad \varepsilon_6 := 2\varepsilon_{12}, \qquad i = 1, 2, 3 \tag{6}$$

and analogously for σ. In this notation the most general linear relation between stress and strain can be written as

$$\sigma_i = \sum_{k=1}^{6} \Gamma_{ik}\varepsilon_k, \qquad i = 1, \dots, 6, \tag{7}$$

Γ_{ik} being the components of a symmetric 6×6 matrix Γ. Without any symmetries the relation is described by $6(6+1)/2 = 21$ independent coefficients. For the following three cases, in which specific symmetries are given by transformations $x_i \mapsto x'_i$, find the number of independent coefficients of Γ.

(i) Inversion: $x_i \mapsto -x_i$.

(ii) Monoclinic system: One twofold axis of rotation: $x \mapsto x' = (-x_1, -x_2, x_3)$.

(iii) Cubic system: Four threefold axes of rotation (diagonals of a cube):

$$(x_1, x_2, x_3) \mapsto (x_2, x_3, x_1), (x_2, -x_3, -x_1), (-x_2, x_3, -x_1), (-x_2, -x_3, x_1) \tag{8}$$

Hints: The tensors ε_{ik}, σ_{ik} transform as $x_i x_k$. Therefore, from the transformation of the coordinates we can find the transformation

$$\hat{R} : (\varepsilon_1, \dots, \varepsilon_6) \mapsto (\varepsilon'_1, \dots, \varepsilon'_6) \tag{9}$$

(the same acts on σ). From this we can derive $\Gamma' = \hat{R}\Gamma\hat{R}^{-1}$. If the transformation $x \mapsto x'$ is a symmetry we require $\Gamma' = \Gamma$. For the third case use a combination of transformations of (8) to conclude that $x \mapsto x' = (-x_1, -x_2, x_3)$ is also a symmetry transformation as in the second case and therefore the coefficients vanishing in the second case also vanish in the third case. Then look at the first of (8) and find the permutation of indices in the mapping (9) . From $\Gamma' = \hat{R}\Gamma\hat{R}^{-1}$ conclude that both indices of Γ_{ij} permute in the same way and from this derive the form of Γ.