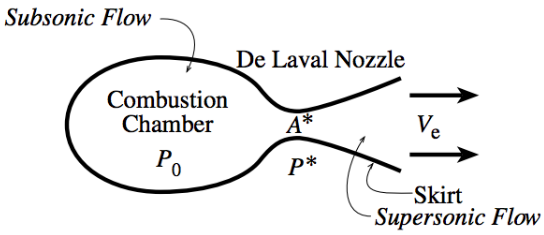
Exercise 1. ***Rocket Thrust***

Consider the *de Laval Nozzle* as depicted below. These nozzles are used to accelerate a gas to supersonic speed and then shape the exhaust to convert the heat energy to directed kinetic energy in an efficient manner and are in rocket engines and supersonic jets, for example.



To analyze this system we will start by examining transitions from subsonic to supersonic flow restricting ourselves to stationary, irrotational flow, and assume viscosity and gravity are negligible. The pressure can be regarded as a function of the density alone $P = P(\rho)$. Under these assumptions, consider a narrow bundle of streamlinescalled a *streamtube* and introduce A: its cross sectional area normal to the flow.

(a) Use the conservation of mass equation for a compressible flow to derive a relation between the rate at which mass passes through the streamtube's cross section and its position alone the tube. Write this relation in differential form.

(b) Define stationary and irrotational mathematically. Next use the conservation of energy equation for a compressible flow to show that it reduces to the first Bernoulli's theorem under our assumptions.

(c) Use the first law of thermodynamics and the fact the flow is adiabatic to rewrite the Bernoulli equation, the result from part b) in differential form relating the density, velocity and sound speed.

Hint. Recall $dh = \frac{dP}{\rho} + Tds$

(d) Combine your results from a) and c) to derive two equations, one for $\frac{dv}{v}$ and one for $\frac{d\rho}{\rho}$ relating each to the streamtube's cross section. Introduce a value $M \equiv v/c$ known as the *Mach number.*

(e) How does v behave with A in the following regimes?

(i) Flow is extremely subsonic

(ii) Flow is barely subsonic

(iii) Flow is supersonic

(iv) Flow is extremely supersonic

(f) Sketch the variation of the cross sectional area of the streamtube vs. Mach number. Where can the transition from subsonic to supersonic flow occur?

(g) Assume we have a de Laval nozzle with $A(x) = \frac{1}{x} + x$ and we are working with a perfect gas with $\frac{P}{P_0} = \frac{\rho}{\rho_0}^\gamma$ $\gamma = 7/5$. Solve for x/x_s as a function of M and plot your solution. Assume the throat is located at $x = 1$.

Exercise 2. ***Adiabatic, Spherical Accretion of Gas onto a Black Hole***

Consider a black hole or neutron star with mass M at rest in interstellar gas that has a constant ratio of specific heat γ. This exercise will show how gravity can play a role analogous to a de Laval nozzle, triggering the transition of the flow from subsonic to supersonic. This problem was first solved by Bondi in 1952.

(a) let ρ_∞ and c_∞ be the density and sound speed in the gas far from the black hole or neutron star. Use dimensional analysis to estimate the rate of accretion of mass M onto the black hole in terms of the parameters of the system and the gravitation constant G.

(b) Show that

$$(v^2 - c^2)\frac{1}{\rho}\frac{d\rho}{dr} = \frac{GM}{r^2} - \frac{2v^2}{r}. \tag{1}$$

(c) Derive the relation between flow speed v_s, sound speed c_s and radius r_s at the *sonic point*, the transition point for subsonic to supersonic flow. Show they are related by

$$v_s^2 = c_s^2 = \frac{GM}{2r_s} \tag{2}$$

(d) Use the result of part c) combined with the Bernoulli equation, this time with the effects of gravity included, to deduce that the sound speed at the sonic point is related to that at infinity by

$$c_s^2 = \frac{2c_\infty^2}{5 - 3\gamma} \tag{3}$$

and that the radius of the sonic point is

$$r_s = \frac{(5 - 3\gamma)}{4}\frac{GM}{c_\infty^2} \tag{4}$$

(e) Derive a precise value for the mass accretion rate. Compare to your estimate in part a).

(f) Much of the ISM is hot and ionized. Assuming a density of one proton per cm^3 and $T = 10^4$K, what is the mass accretion rate on to a 10 solar mass black hole. How long does it take for the black hole's mass to double?