ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Quantum Mechanics II
Series 3.

FS 2013
A. Signer

Exercise 1.   *Zeeman effect*

In the following exercise we study the effect of an external magnetic field on the energy levels of an hydrogen-like atom, the so-called Zeeman effect.

The interaction hamiltonian can be written:

$$H_Z = -(\vec{\mu}_l + \vec{\mu}_s) \cdot \vec{B} = \frac{e}{2m}(\vec{L} + 2\vec{S}) \cdot \vec{B} \tag{1}$$

The nature of the Zeeman effect depends on the strength of the external magnetic field, compared to the internal magnetic field which gives rise to the fine structure splitting. We analyze the two different regimes separately.

(a) **Weak-field**

If $B_{ext} \ll B_{int}$ the fine-structure dominates. Show that the energy shift due to the interaction with the magnetic field can be written as:

$$\Delta E_Z = \frac{e}{2m}\vec{B}_{ext} \cdot \langle \vec{L} + 2\vec{S} \rangle = \frac{e}{2m} g_J \vec{B}_{ext} \cdot \langle \vec{J} \rangle$$

where

$$g_J = \left[1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2\,j\,(j+1)} \right]$$

is the **Landé g-factor**.

Assuming that $\vec{B}_{ext} = (0, 0, B_z)$, draw the energy-splitting of the levels $2P_{1/2}$ and $2P_{3/2}$ due to the Zeeman effect.

Hints

- Since the magnetic field can be considered as a perturbation of the fine structure, the "good" quantum numbers to describe the energy splitting are n, l, j, m_j.
- In order to compute the mean value of the Spin operator $\langle \vec{S} \rangle$ you can use the Wigner-Eckart theorem, namely:

$$\langle \vec{S} \rangle = \left\langle \frac{\vec{S} \cdot \vec{J}}{J^2}\, \vec{J} \right\rangle.$$

(b) **Strong-field (Paschen-Back effect)**

On the other hand, if $B_{ext} \gg B_{int}$ the Zeeman effect dominates.

Putting again $\vec{B}_{ext} = (0, 0, B_z)$, draw the energy-splitting for the level $2P$ in this regime neglecting the fine structure contribution and compare it with the one above.

Hints

- Following what has been done in the weak case and noticing that in this regime the relative importance of the Zeeman effect and of the spin-orbit is inverted, choose the right basis of eigenvectors to compute the energy shifts in first order perturbation theory.

Exercise 2. *Spin-statistics*

(a) Positronium is the bound state of an electron-positron pair. The wave function for the ground state with $L = 0$ (zero orbital angular momentum) can be decomposed into the tensor product of the spinorial and spatial eigenfunctions:

$$\Psi_{e^+e^-}(\vec{x}_1, \vec{x}_2) = \chi_S \otimes \psi(\vec{x}_1, \vec{x}_2). \tag{2}$$

Defining $\vec{S} = \vec{S}_{e^+} + \vec{S}_{e^-}$ and knowing that e^+ and e^- have spin 1/2, write down the spinorial eigenfunctions for the ground states of para-positronium (spin 0, $^1S_0$) and ortho-positronium (spin=1, $^3S_1$) in the basis $|S_{z\,e^-}, S_{z\,e^+}\rangle$.

Recalling the symmetry properties of the wavefunction of a multi-fermion state, discuss the symmetry of the spatial wavefunction under exchange of the two fermions.

(b) The particle Δ^{++} is a bound state of three identical spin-1/2 particles called up-quarks.

We can imagine to decompose the total wavefunction for this bound state as

$$\Psi_{\Delta^{++}} = \chi_{spin}\,\Psi_{space}. \tag{3}$$

The total spin of Δ^{++} is 3/2 and in the ground state $L = 0$ the spatial wavefunction is symmetric . Discuss why the total wavefunction is not acceptable for a bound state of three identical fermions.

We can introduce a new quantum number c, called *colour*, with three possible eigenvalues associated to the eigenfunctions $|r\rangle, |g\rangle, |b\rangle$. Write down the total wave function for the colour of the ground state as linear combination of the eigenstates above, in such a way that the total wavefunction respects the expected symmetry properties.

Exercise 3. *The Fermi gas*

Let us consider a system of a very large number of non-interacting spin-1/2 particles contained in a box. If the box is large enough, the properties of the system are independent of the shape of the box, and the system is called a *Fermi gas*. Since the particles are non-interacting, each of them satisfies the free-particle Schroedinger equation

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi(x, y, z) = E\psi(x, y, z) \tag{4}$$

We consider impenetrable walls with boudary conditions $\psi = 0$. For a large cubic box of side L the wavefunction is given by

$$\psi_{n_x n_y n_z} = C \sin\left(\frac{n_x\pi}{L}x\right)\sin\left(\frac{n_y\pi}{L}y\right)\sin\left(\frac{n_z\pi}{L}z\right) \tag{5}$$

where $C = (8/L^3)^{1/2}$ and n_x, n_y, n_z are positive integers. The corresponding eigenvalues are

$$E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2 \qquad n^2 = n_x^2 + n_y^2 + n_z^2 \tag{6}$$

The total one-particle wavefunction is the so-called spin-orbital

$$\psi_{n_x n_y n_z m_s} = \psi_{n_x n_y n_z}(x, y, z)\chi_{\frac{1}{2}, m_s} \tag{7}$$

where χ is the spin wavefunction with $m_s = \pm 1/2$.

Since the energy spacings are very small for a macroscopic box, it is a good approximation to consider that the energy levels are distributed nearly continuously. We may then introduce the density of states $D(E)$, which is defined in such a way that $D(E)dE$ is the number of states with energy in the range $(E, E + dE)$. In the space formed by the axes n_x, n_y, n_z we are interested in the octant for which $n_x > 0, n_y > 0, n_z > 0$. Each state is associated to a point (n_x, n_y, n_z) of a cubical lattice, and every elementary cube of the lattice has unit volume. Hence, the total number of states up to an energy E is well approximated by the volume of the octant of a sphere of radius $n = (n_x^2 + n_y^2 + n_z^2)^{1/2}$. The total number of individual particle states for energies up to E is therefore

$$N_s = 2\frac{1}{8}\frac{4}{3}\pi \, n^3 \tag{8}$$

(a) Starting from N_s, and setting $V = L^3$, derive the density of states $D(E)$. Describe schematically how the energy levels are populated by N identical spin-1/2 particles in the ground state of the Fermi gas at absolute temperature $T = 0$, up to an energy E_F called the *Fermi energy*.

(b) Derive the explicit value of E_F for a gas containing N particles.

(c) Compute the total energy of the gas and the average particle energy.