Exercise 1. ***Interaction picture***

The interaction picture in quantum mechanics is an intermediate representation between the Schroedinger picture and the Heisenberg picture.

Consider the Schoedinger problem

$$i\,\hbar\,\partial_t\,|\psi(t)\rangle = H(t)|\psi(t)\rangle,$$

with the hamiltonian:

$$H(t) = H_0 + V(t).$$

Assuming that H_0 is an exactly solvable time-independent hamiltonian, the *Interaction Picture* is defined as:

$$|\psi_I(t)\rangle = U_0(t_0,t)|\psi(t)\rangle \tag{1}$$

$$\mathcal{A}_I(t) = U_0(t_0,t)\,\mathcal{A}\,U(t,t_0), \tag{2}$$

where the evolution operator of H_0 reads:

$$U_0(t,t_0) = \mathrm{e}^{iH_0(t-t_0)/\hbar}.$$

(a) Show *by explicit computation* that the time evolution operator in the interaction picture can be written as

$$U_I(t,t_1) = U_0(t_0,t)U(t,t_1)U_0(t_1,t_0) \tag{3}$$

in agreement with (2).

(b) Starting from the formula above show that

$$i\,\hbar\,\partial_t U_I(t,t_0) = V_I(t)\,U_I(t,t_0),$$

where $V_I(t)$ is just the potential $V(t)$ in the interaction picture.

Exercise 2. ***Long-distance scattering***

Starting from the form of the wave function for $r \to \infty$

$$\psi(\vec{r}\,) = \mathrm{e}^{i\vec{k}\cdot\vec{r}} + f(\theta)\frac{\mathrm{e}^{ik\,r}}{r}, \tag{4}$$

(a) Compute the probability density of the outgoing spherical plane wave:

$$\vec{j} = \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$$

(b) Show that the asymptotic solution in (4) indeed solves the Schroedinger equation in the limit $r \to \infty$

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\psi(\vec{r}\,) = E\psi(\vec{r}\,)$$

as long as the scattering potential satisfies

$$\lim_{r\to\infty} r\,V(r) \to 0.$$

Exercise 3. ***Elastic-Scattering from a central potential***

Consider the elastic scattering off a central potential

$$V(r) = \frac{\epsilon}{r^2}, \qquad \text{with} \qquad \epsilon \ll \frac{\hbar^2}{2m}.$$

The Schroedinger equation reads as usual:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{\epsilon}{r^2}\right)\psi(\vec{r}) = E\psi(\vec{r}). \tag{5}$$

Using the usual partial-wave decomposition:

$$\psi(\vec{r}) = \sum_{l=0}^{\infty} R_l(r) P_l(\cos\theta),$$

compute the phase shifts $\delta_l(k)$ and the scattering amplitude $f(\theta)$,

Hint:

- Recall that:

$$\sum_{l=0}^{\infty} P_l(\cos\theta) = \frac{1}{2\sin\theta/2}.$$