Exercise 1. ***Optical Theorem***

Writing the relation between the S-matrix and the T-matrix as

$$S_{\beta\alpha} = \delta(\beta - \alpha) - i\,2\pi\,\delta(E_\alpha - E_\beta)\,T_{\beta\alpha}, \tag{1}$$

the unitarity condition on the S-matrix reads

$$(S^\dagger S)_{\beta\alpha} = \sum_\gamma (S_{\gamma\beta})^*(S_{\gamma\alpha}) = \delta(\beta - \alpha),$$

with a sum over all intermediate states γ including an integration over their momenta.

(a) Show that the unitarity condition for $\beta = \alpha$ implies

$$-2\,\mathrm{Im}(T_{\alpha\alpha}) = (2\pi)\sum_\gamma \delta(E_\gamma - E_\alpha)\,|T_{\gamma\alpha}|^2 \tag{2}$$

(b) Starting from the equation above, prove the Optical Theorem:

$$\sigma_{\mathrm{tot}} = \frac{4\pi}{k_\alpha}\mathrm{Im}(f_{\alpha\alpha}) \tag{3}$$

where $f_{\alpha\alpha} = -\frac{m_\alpha}{2\pi\hbar^2}T_{\alpha\alpha}$ is the elastic scattering amplitude in the forward direction.
Hint. Recall that the differential cross-section for the process $\alpha \to \gamma$ can be written as:

$$\frac{d\sigma_{\alpha\to\gamma}}{d\Omega} = \frac{1}{\mathit{flux}}\,\frac{2\pi}{\hbar}\,|T_{\gamma\alpha}|^2\,\frac{d\rho}{d\Omega},$$

where ρ is the density of states

$$\rho = \int \frac{d^3\vec{k}_\gamma}{(2\pi)^3}\,\delta(E_\gamma - E_\alpha)$$

Exercise 2. ***The photoelectric effect***

We want to treat the photoelectric effect using the quantization of the electromagnetic field. A photon is absorbed by the atom and an electron is excited from the bound state:

$$\psi_i(\vec{r}) = \sqrt{\frac{Z^3}{\pi a_0^3}}\,e^{-\frac{Zr}{a_0}}\ , \tag{4}$$

where $a_0 = \frac{\hbar^2}{me^2}$, to a free state

$$\psi_f(\vec{r}) = e^{i\vec{k}_e\cdot\vec{r}}\ . \tag{5}$$

(a) Compute the transition matrix element $V_{fi} = \langle\psi_f; (n-1)(\vec{k}_\lambda, \lambda)|V|\psi_i; n(\vec{k}_\lambda, \lambda)\rangle$.

(b) Using the Fermi's golden rule for a continuum of states, show that the differential cross section is given by:

$$\frac{d\sigma}{d\Omega} = \frac{e^2}{\hbar c}\frac{32\,\hbar\, k_e}{m\,\omega_\lambda}(\vec{\varepsilon}_\lambda \cdot \vec{k}_e)^2 \frac{\left(\frac{Z}{a_0}\right)^5}{\left[\left(\frac{Z}{a_0}\right)^2 + |\vec{k}_\lambda - \vec{k}_e|^2\right]^2} , \tag{6}$$

where $\vec{\varepsilon}_\lambda, \vec{k}_\lambda, \omega_\lambda$ corresponds to the absorbed photon described by $(\vec{k}_\lambda, \lambda)$.

(c) If the photon energy is large compared to the binding energy of the electron but small compared with the rest mass energy ($W = \frac{mZ^2e^4}{2\hbar^2} = \frac{\hbar^2}{2m}\frac{Z^2}{a_0^2} \ll \hbar\omega_\lambda \ll mc^2$), show that for an unpolarized light (equal polarization along x and y direction) the scattering cross section can be written as

$$\frac{d\sigma}{d\Omega} = \frac{e^2}{\hbar c} 32 \left(\frac{k_e\, a_0}{Z}\right)^3 \left(\frac{W}{\hbar\omega_\lambda}\right)^5 \frac{\left(\frac{a_0}{Z}\right)^2 \sin^2\theta}{\left(1 - \frac{v}{c}\cos\theta\right)^4} , \tag{7}$$

where $v = \frac{\hbar k_e}{m}$ is the electron velocity, the z-axis coincides with the direction of the incident photon and the photon velocity is described by the polar angle θ ($\vec{k}_\lambda \cdot \vec{k}_e = k_\lambda k_e \cos\theta$).