Exercise 1. *Dirac Matrices*

(a) Prove that any matrices $\vec{\alpha}$, β satisfying

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij}; \quad \beta^2 = 1; \quad \{\beta, \alpha^i\} = 0;$$

are traceless with eigenvalues ± 1.

(b) Using the properties above, argue that they must be even dimensional.

(c) What is the minimum dimensionality of the Dirac matrices? Can they be 2×2 matrices?

Exercise 2. *Relativistic Hydrogen Atom*

In non-relativistic quantum mechanics the radial Schroedinger equation for the Hydrogen atom reads

$$\left[\frac{\hbar^2}{2m}\left(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2}\right) - \frac{Z\alpha\,\hbar c}{r} - E\right] R(r) = 0 \tag{1}$$

The quantisation condition requires that $n' = n - (l+1)$ must be a non-negative integer, and the energy levels are

$$E = -\frac{m\,c^2\,(Z\alpha)^2}{2n^2}; \qquad n = 1, 2, \ldots$$

where $\alpha = e^2/(\hbar c)$. Consider now the Dirac equation for an electron interacting with an external electromagnetic field

$$\left(i\hbar\,\gamma^\mu \partial_\mu + \frac{e}{c}\,\gamma^\mu A_\mu - mc\right)\psi = 0 \tag{2}$$

(a) Show that by multiplying Eq.(2) by the operator $(i\hbar\gamma^\mu\partial_\mu + e/c\,\gamma^\mu A_\mu + mc)$ we get the equation

$$\left[\left(i\hbar\,\partial\!\!\!/ + \frac{e}{c}\,A\!\!\!/\right)^2 + \frac{e\hbar}{2c}\sigma^{\mu\nu}F_{\mu\nu} - m^2c^2\right]\psi = 0 \tag{3}$$

where $\sigma^{\mu\nu} = (i/2)\,[\gamma^\mu, \gamma^\nu]$ and $F_{\mu\nu}$ is the usual field strength tensor.

(b) In the case of a pure Coulomb interaction the only non-vanishing component of the potential is $A_0 = Ze/r$. It is convenient to work in the so called chiral representation for the gamma matrices where

$$\sigma^{0j} = i\begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

Show that Eq.(3) can be formally written as two decoupled Schroedinger-like equations for the two spinor components $\psi_\pm$

$$\left[\hbar^2\left(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1) - Z^2\alpha^2 \mp iZ\alpha\,\vec{\sigma}\cdot\hat{r}}{r^2}\right) - \frac{Z\alpha\,2E\,\hbar}{c\,r} - \frac{E^2 - (m\,c^2)^2}{c^2}\right]\psi_\pm = 0 \tag{4}$$

(c) In order to write the equivalent of the radial equation Eq.(1), we decompose the spinor wavefunction as follows

$$|\psi_\pm(\vec{r})\rangle = R_\pm(r)|j\,m\,l\,\rangle$$

where $|\,j\,m\,l\,\rangle$ are the generalisation of the spherical harmonics and $R_\pm$ are the radial wave functions.

Note that, since the operator $\tilde{L}^2 = L^2 - \hbar^2 Z^2\alpha^2 \mp i\hbar^2 Z\alpha\,\vec{\sigma}\cdot\hat{r}$ commutes with the total angular momentum $\vec{J} = \vec{L} + \vec{\sigma}/2$, we can look at the subspace of the Hilbert space with j fixed, in such a way that l can take two values:

$$l_\pm = j \pm \frac{1}{2},$$

and obviously $L^2|j\,m\,l\,\rangle = \hbar^2 l(l+1)\,|j\,m\,l\,\rangle$.

Show that with an appropriate phase-choice we have (for j and m fixed):

$$\langle j\,m\,l_\pm\,|\vec{\sigma}\cdot\hat{r}|j\,m\,l_\pm\,\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

i.e., $\vec{\sigma}\cdot\hat{r}$ switches $l_\pm \to l_\mp$.

(d) Using the results above show that Eq. (4) can be rewritten as an equation for the radial wave functions $R_\pm$ of the same form of (1):

$$\left[\frac{\hbar^2}{2m}\left(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{\lambda(\lambda+1)}{r^2}\right) - \frac{Z\,\tilde{\alpha}\,\hbar c}{r} - \tilde{E}\right] R_\pm(r) = 0 \tag{6}$$

with

$$\tilde{\alpha} = \alpha\frac{E}{mc^2}, \qquad \tilde{E} = \frac{E^2 - (mc^2)^2}{2\,mc^2},$$

and the two eigenvalues

$$\lambda = \sqrt{(j+1/2)^2 - Z^2\alpha^2} \qquad \text{and} \qquad \lambda = \sqrt{(j+1/2)^2 - Z^2\alpha^2} - 1\,,$$

which can be rewritten as

$$\lambda = (j \pm 1/2) - \delta_j, \quad \text{with} \quad \delta_j = j + 1/2 - \sqrt{(j+1/2)^2 - Z^2\alpha^2}.$$

(e) Finally, with these identifications, show that the energy eigenvalues in Dirac case can be written as:

$$E_{nj} = \frac{mc^2}{\sqrt{1 + Z\alpha^2/(n-\delta_j)^2}}, \tag{7}$$

and compare the result with the *fine-structure* corrections to the non-relativistic result at $\approx \mathcal{O}(\alpha^4)$.