Exercise 1. *Transformation of $\psi(x)$*

Under a Lorentz transformation Λ the solution to the Dirac equation transforms as $\psi(x) \to \psi'(x') = S(\Lambda)\psi(x)$, where the matrix $S(\Lambda)$ has to satisfy

$$\Lambda^\mu{}_\nu \gamma^\nu = S^{-1}(\Lambda)\gamma^\mu S(\Lambda)$$

(a) Show that for an infinitesimal proper Lorentz transformation with parameters $\omega_{\mu\nu}$ we have $S(\Lambda) = 1 + \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}$.

(b) Find the transformation of the Dirac adjoint $\overline{\psi}(x)$ under Lorentz transformations.

(c) Find the explicit form of $S(\Lambda)$ in the Dirac representation for Λ being a rotation about the z-axis through an angle ϕ.

Exercise 2. *Hartree-Fock*

Consider a system of N electrons described by:

$$H = T + U + V = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i=1}^{N} \frac{-Ze^2}{|r_i|} + \sum_{i>j}^{N} \frac{e^2}{|r_i - r_j|} \ . \tag{1}$$

The fermionic state ψ_i is defined in the second quantization by $\hat{a}_i^\dagger |0\rangle = |i\rangle = |\psi_i\rangle$, where the creation and annihilation operators satisfy the usual relation

$$\{\hat{a}_i, \hat{a}_j\} = \{\hat{a}_i^\dagger, \hat{a}_j^\dagger\} = 0 \quad \text{and} \quad \{\hat{a}_i^\dagger, \hat{a}_j\} = \delta_{i,j} \ . \tag{2}$$

(a) Show that the Hamiltonian can be written as:

$$H = \sum_{ij} \langle i|T|j\rangle \, \hat{a}_i^\dagger \hat{a}_j + \sum_{ij} \langle i|U|j\rangle \, \hat{a}_i^\dagger \hat{a}_j + \frac{1}{2}\sum_{ijkl} \langle i,j|V|k,m\rangle \, \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_m \hat{a}_k \ . \tag{3}$$

Hint. By convention $|i,j\rangle = \hat{a}_i^\dagger \hat{a}_j^\dagger |0\rangle$.

(b) We define a state $|\Psi\rangle = \hat{a}_1^\dagger \cdots \hat{a}_N^\dagger |0\rangle$. Show that this state is antisymmetric.

(c) Show that

$$\langle\Psi|\hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_m|\Psi\rangle = (\delta_{im}\delta_{jk} - \delta_{ik}\delta_{jm}) \langle\Psi|\hat{a}_m^\dagger \hat{a}_k^\dagger \hat{a}_k \hat{a}_m|\Psi\rangle \ . \tag{4}$$

(d) For the electrons the state is defined by an orbital part ψ_i and an spin part s_i. Assuming the ground state is given in the form of Ψ, show that the energy of this state is:

$$\begin{aligned}\langle\Psi|H|\Psi\rangle &= \sum_{i=1}^{N} \int d^3\vec{r} \left(-\frac{\hbar^2 |\nabla\psi_i(\vec{r})|^2}{2m} - \frac{Ze^2}{|\vec{r}|} |\psi_i(\vec{r})|^2 \right) \\ &+ \frac{1}{2}\sum_{i,j=1}^{N} \int d^3\vec{r}_1 \int d^3\vec{r}_2 \frac{e^2}{|r_1 - r_2|} \left(|\psi_i(\vec{r}_1)|^2 |\psi_j(\vec{r}_2)|^2 - \delta_{s_i,s_j} \psi_i^*(\vec{r}_1)\psi_i(\vec{r}_2)\psi_j^*(\vec{r}_2)\psi_j(\vec{r}_1) \right) \ .\end{aligned} \tag{5}$$

Compare the result with the one known from the Hartree-Fock theory.