Exercise 7.1 Lindhard function

In the lecture it was shown how to derive the dynamical linear response function $\chi_0(\boldsymbol{q},\omega)$ which is also known as the Lindhard function:

$$\chi_0(\boldsymbol{q},\omega)=\frac{1}{\Omega}\sum_{\boldsymbol{k}}\frac{n_F(\epsilon_{\boldsymbol{k}+\boldsymbol{q}})-n_F(\epsilon_{\boldsymbol{k}})}{\epsilon_{\boldsymbol{k}+\boldsymbol{q}}-\epsilon_{\boldsymbol{k}}-\hbar\omega-i\hbar\eta}. \tag{1}$$

Calculate the static Lindhard function $\chi_0(\boldsymbol{q})$ of free electrons for the 1 and 3 dimensional cases at $T=0$.
Hint: We are only interested in the real part of $\chi_0(\boldsymbol{q},\omega)$. You may use the equation $\lim_{\eta\to 0}(z-i\eta)^{-1}=\mathcal{P}\left(1/z\right)+i\pi\delta(z)$. Notice that in 3 dimensions we can choose $\boldsymbol{q}=q\boldsymbol{e}_z$ to point in the z-direction due to the isotropy of a system of free electrons. Also, changing to cylindrical coordinates in order to calculate the integral turns out to be helpful.

Exercise 7.2 Zero-sound excitations

The dispersion relation of the plasmon excitation is finite for all $\boldsymbol{q}$'s. This appearance of a finite excitation energy is a consequence of the long range character of the Coulomb potential $V_{\text{Coulomb}}(\boldsymbol{r})=e^2/|\boldsymbol{r}|$. A system consisting of fermions with a purely local potential

$$V_{\text{local}}(\boldsymbol{r})=U\cdot\delta(\boldsymbol{r}) \tag{2}$$

shows a different behaviour at $\boldsymbol{q}=0$. In this exercise we will follow sections (3.2.1) to (3.2.3) of the lecture notes in order to discuss the plasmon excitation in the context of a local interaction.

a) As a warm-up, derive the relation between the particle distribution $\delta n(\boldsymbol{r},t)$ and its induced potential $V_{\text{ind}}(\boldsymbol{r},t)$ in the $(\boldsymbol{k},\omega)$-space.

b) Find the imaginary part of the response function $\chi(\boldsymbol{q},\omega)$ for small $\boldsymbol{q}$'s. What is the dispersion relation to lowest order in $\boldsymbol{q}$?

c) The upper boundary line of the particle-hole continuum is given by

$$\hbar\omega_{q,\max}=\frac{\hbar^2}{2m}\left(q^2+2k_Fq\right)=\frac{\hbar^2q^2}{2m}+\hbar v_Fq, \tag{3}$$

where v_F is the Fermi velocity and $q=|\boldsymbol{q}|$. What is the condition on U in order to obtain stable plasmon excitations (quasi-particles)?

This collective mode was first predicted by Landau in 1957 in the framework of his theory of Fermi liquids. In 1966 zero sound was experimentally observed in He^3 by Abel, Anderson and Wheatley. References :

- L. D. Landau, JETP 32, 59 (1957), Soviet Phys. JETP 5, 101 (1957).
- W. R. Abel, A. C. Anderson, and J. C. Wheatley, Propagation of Zero Sound in Liquid He^3 at Low Temperatures, Phys. Rev. Lett. 17, 7478 (1966).
- L. P. Pitaevskii, Zero Sound in Liquid He^3, Sov. Phys. Usp. , **10, 100 (1967).**

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