Solid State Theory
Exercise 9

Exercise 9.1 Landau Diamagnetism

Calculate the orbital part of the magnetization (ignoring the Zeeman term) of the free electron gas in 3D in the limit of low temperature and small external field ($T \to 0,\ B \to 0$). In addition, show that the magnetic susceptibility at $T = 0$ and $B = 0$ is given by

$$\chi = -\frac{1}{3}\frac{m^2}{m^{*2}}\chi_P, \tag{1}$$

where χ_P is the Pauli susceptibility, which is given by $\chi_P = \mu_B^2 \rho(\epsilon_F)$ (at $T = 0$), where $\mu_B = \frac{e\hbar}{2mc}$, and ρ is the density of states (including spin degeneracy).
Hint: Calculate the free energy

$$F = N\mu - k_B T \sum_i \ln\left[1 + e^{-(\epsilon_i - \mu)/k_B T}\right] \tag{2}$$

at $T = 0$ to second order in B using the Euler-Maclaurin formula,

$$\sum_0^{n_0} f(n) \approx \left(\int_{-1/2}^{n_0+1/2} f(n)\,\mathrm{d}n\right) - \frac{1}{24}\left[f'(n_0 + 1/2) - f'(-1/2)\right]. \tag{3}$$

Exercise 9.2 The lowest Landau level in the Corbino geometry

The Hamilton operator for one electron ($e < 0$) restricted to the plane $z = 0$ and exposed to a magnetic field is given by

$$H = \frac{1}{2m^*}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 + U(r) \tag{4}$$

where $r^2 = x^2 + y^2$. The annular potential

$$U(r) = \frac{C_1}{r^2} + C_2 r^2 \tag{5}$$

with $C_1, C_2 > 0$ yields a Corbino (after the Italian physicist O. M. Corbino, 1876-1937) geometry confining the electron on a two-dimensional ring. Let the magnetic field $\boldsymbol{B}$ be homogeneous and directed along the z-axis for $r > 0$. In addition, there is a magnetic flux $\Phi = \nu\Phi_0$ through the origin ($r = 0$) that does not physically touch the electron:

$$\boldsymbol{B} = [B + \nu\Phi_0\,\delta(\boldsymbol{r})]\,\boldsymbol{e}_z, \tag{6}$$

with $B, \nu > 0$. $\Phi_0 = hc/|e| = 2\pi\hbar c/|e|$ is the magnetic flux quantum.

a) Show that the vector potential can be chosen in the symmetric gauge

$$\boldsymbol{A} = \frac{1}{2}\left(B + \frac{\nu\Phi_0}{\pi r^2}\right)(x\boldsymbol{e}_y - y\boldsymbol{e}_x). \tag{7}$$

b) We now solve this single-particle problem in the symmetric gauge. First, we consider $\nu = 0$ and use the following ansatz for the wave functions of the lowest Landau level:

$$\psi_m(r, \phi) = A r^\alpha e^{-im\phi} e^{-\frac{r^2}{4l^{*2}}}. \tag{8}$$

In order to avoid a singularity at the origin we have to demand that $\alpha \geq 0$. The uniqueness of the wave function is ensured by $m \in \mathbb{Z}$.
Show that the following Schrödinger equation for the radial part is obtained:

$$\left\{ \frac{\hbar^2}{2m^*} \left[-\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \left(\frac{m}{r} - \frac{r}{2l^2} \right)^2 \right] + U(r) - E_m \right\} r^\alpha e^{-\frac{r^2}{4l^{*2}}} = 0, \tag{9}$$

where $l^2 = \hbar c / |eB|$.
Hint: $L_z = xp_y - yp_x = -i\hbar\partial_\phi$ and $\partial_x^2 + \partial_y^2 = (1/r)\partial_r r \partial_r + (1/r)^2 \partial_\phi^2$.
Also, show that the following relations are obtained:

$$\alpha = \sqrt{m^2 + C_1^*}, \qquad \frac{1}{l^{*2}} = \frac{1}{l^2}\sqrt{1 + C_2^*}, \tag{10}$$

where $C_1^* = 2m^* C_1 / \hbar^2$ and $C_2^* = 8l^4 m^* C_2 / \hbar^2$ are dimensionless parameters.
Finally, show that the energy (with $\omega_c = |eB|/m^* c$) is given by

$$E_m = \frac{\hbar\omega_c}{2} \left[\frac{l^2}{l^{*2}}(\alpha + 1) - m \right]. \tag{11}$$

c) i) Consider the case $U \equiv 0$. Plot the radial part of the wave function for several m and show that it has a maximum at $r_m = \sqrt{2m}\, l$. Thus, the wave functions are localized on circles of radius r_m. Compute the magnetic flux penetrating the circle of radius r_m. How big is the degeneracy of the lowest Landau level?

ii) Show that for $C_1^*, C_2^* \ll 1$ and $m \gg 1$ the energy in Eq. (11) is approximately

$$E_m \approx \hbar\omega_c/2 + U(r_m).$$

Thus, the wave functions are localized on curves of equal potential energy.

d) Compute the angular speed ω_m in the state ψ_m given by

$$\omega_m := \langle \psi_m | v_\phi / r | \psi_m \rangle = -\frac{1}{\hbar}\frac{\partial E_m}{\partial m}, \tag{12}$$

and show that under the conditions stated in c) ii) one finds

$$\omega_m \approx -\frac{U'(r_m)}{m^* \omega_c r_m}. \tag{13}$$

Why are the states located on different boundaries of the Corbino disk called *chiral* edge states?

e) We assume now $\nu > 0$ and use the same ansatz as above for the wave functions of the lowest Landau level. Show that the resulting Schrödinger equation is equal to Eq. (9) except for the substitution $m \to m - \nu$.
Use gauge transformations of the vector potential to show that under certain conditions the contribution from the magnetic flux through the origin can be transformed away. What is the condition on ν?

f) What happens if the magnetic flux through the central hole is turned on adiabatically from $\nu = 0$ to $\nu = 1$? Use the above findings to recapitulate Laughlin's gauge argument for the quantization of the Hall conductivity (see Ch. 4.2.2 in the lecture notes). What is the role of the additional magnetic flux?

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