Solid State Theory
Exercise 11

Exercise 11.1 Relaxation time approximation

In this exercise we will show that the so-called single-relaxation-time approximation,

$$\left(\frac{\partial f(\boldsymbol{k})}{\partial t}\right)_{\text{coll}} = -\int \frac{\mathrm{d}^d k'}{(2\pi)^d} W(\boldsymbol{k}, \boldsymbol{k}')[f(\boldsymbol{k}) - f(\boldsymbol{k}')] \quad \longrightarrow \quad -\frac{f(\boldsymbol{k}) - f_0(\boldsymbol{k})}{\tau}, \tag{1}$$

is a true solution to the Boltzmann equation under certain conditions.
We consider a spatially homogeneous two-dimensional metal with an isotropic Fermi surface ($\varepsilon_{\boldsymbol{k}} = \hbar^2 k^2/2m$) at zero temperature. The impurity scattering responsible for a finite resistivity is described by a delta potential in real space,

$$V_{\text{imp}}(\boldsymbol{r}) = V_0\,\delta(\boldsymbol{r}). \tag{2}$$

The system is subject to a homogeneous and time-independent electric field along the x-axis.

a) Show that the transition rates $W(\boldsymbol{k}, \boldsymbol{k}')$ for the impurity potential (2) are constant and non-zero only for scattering events conserving the energy of the incoming state.

b) Write down the static Boltzmann transport equation for this setup in the form

$$\text{“drift-term”} = \text{“collision-integral”} \tag{3}$$

and take advantage of the zero-temperature limit and the symmetries of the system to eliminate all but angular variables.

c) In a case with only angular dependence, it turns out to be useful to expand the drift term $\boldsymbol{\nabla}_{\boldsymbol{k}} f \cdot (e\boldsymbol{E})$ and $\delta f = f - f_0$ in Fourier modes

$$\delta f = \sum_l f_l\, e^{il\varphi}, \qquad \boldsymbol{\nabla}_{\boldsymbol{k}} f \cdot (e\boldsymbol{E}) = \sum_l d_l\, e^{il\varphi}. \tag{4}$$

Rewrite the Boltzmann equation as a set of algebraic equations for the coefficients in the expansion (4)

$$d_m = \sum_n L_{m,n} f_n. \tag{5}$$

d) What are the eigenvalues of the so-called collision operator $L_{m,n}$ and what is their meaning? How can one interpret vanishing eigenvalues?

e) Find a solution to equation (5) and compare δf to the single-relaxation-time approximation, equation (1).

Exercise 11.2 Penetration depth in a superconductor

We consider a superconductor with a normal-conducting component ρ_n and a superconducting component ρ_s, where $\rho = \rho_n + \rho_s$ is the total electron density. The conductivity of the system is given by the conductivity of the two components, $\sigma = \sigma_n + \sigma_s$, with

$$\sigma_n(\omega) = \rho_n \frac{e^2}{m} \frac{\tau}{1 - i\omega\tau} \quad \text{and} \quad \sigma_s(\omega) = i\rho_s \frac{e^2}{m(\omega + i0^+)}. \tag{6}$$

The density of the superconducting component depends on the temperature in the following way (Gorter-Casimir two-fluid model)

$$\rho_s(T) = \rho \left[1 - \left(\frac{T}{T_c} \right)^4 \right]. \tag{7}$$

a) Use the expression for the dielectric constant of a metal $\varepsilon(\omega) = 1 + (4\pi i/\omega)\sigma(\omega)$ in order to compute the penetration depth $\delta(\omega, T)$.

b) Plot the penetration depth $\delta(\omega, T)$ in the limits $T \to T_c$ and $T \to 0$ for small ω as a function of the frequency ω.

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