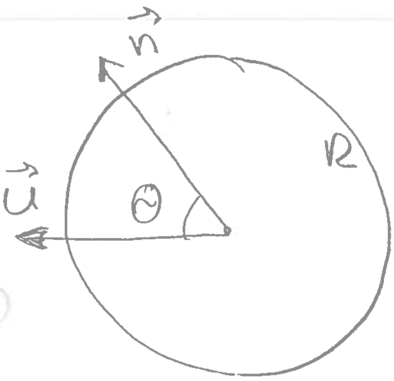
The force acting on the body in potential flow

When the sphere moves through the fluid

$$\varphi = -\frac{R^3(\vec{u}\cdot\vec{n})}{2r^2}, \quad \vec{v} = \frac{R^3}{2r^3}\left[3\vec{n}(\vec{u}\cdot\vec{n}) - \vec{u}\right]$$



For body of an arbitrary shape we also have to solve the Laplace equation

$\nabla^2\varphi = 0$. Far from the body we can do multipole expansion

$$\varphi = \frac{a}{r} + \vec{A}\cdot \operatorname{grad}\frac{1}{r} + \dots$$

The first term $\varphi = \frac{a}{r}$ cannot be present. It corresponds to velocity $v = -a\frac{\vec{r}}{|r|^3}$

If with this velocity field we calculate the flux through the closed sphere with radius R we get

$$\rho \oint \vec{v} \cdot d\vec{S} = 4\pi a \rho$$

But the flux of incompressible fluid through any closed surface should be zero $\Rightarrow a = 0$

Thus the first non-vanishing term at large distances is

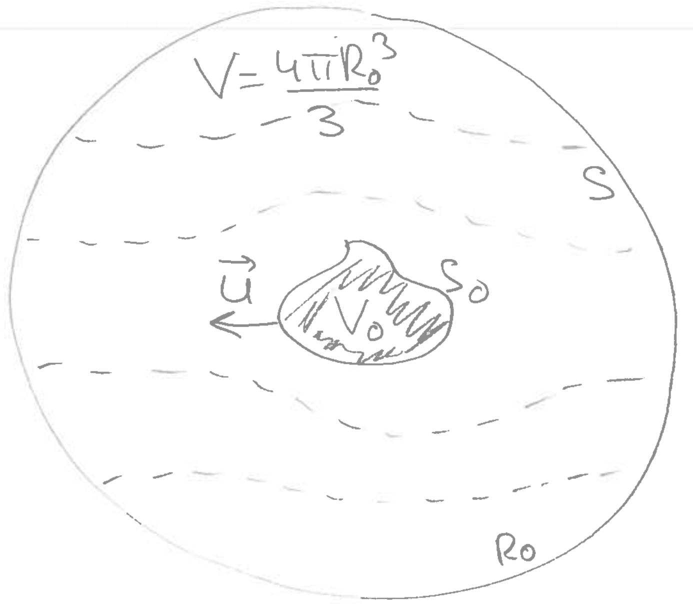
$$\varphi = A \cdot \vec{\nabla}\left(\frac{1}{r}\right) = -\frac{(\vec{A} \cdot \vec{n})}{r^2}$$

$$\vec{v} = \text{grad}\,\varphi = \frac{3[(\vec{A} \cdot \vec{n}) \cdot \vec{n} - \vec{A}]}{r^3}$$

For the sphere $\vec{A} = \frac{\vec{u}\, R^3}{2}$

In general $A_i = \alpha_{ik} u_k$, where the tensor α_{ik} depends on the body shape

Let us calculate the energy $E = \rho \int \frac{v^2}{2} dV$ of the moving fluid outside the body and inside the large sphere of the radius R_0



We write $\vec{v}^2 = \vec{u}^2 + (\vec{v} - \vec{u})(\vec{v} + \vec{u})$

Then $$\int_{r<R_0} v^2 dV = u^2 (V - V_0) + \int (\vec{v} + \vec{u})(\vec{v} - \vec{u}) dV$$

The last integral can be simplified noticing that $\vec{v} + \vec{u} = \nabla(\varphi + \vec{u} \cdot \vec{r})$ and

$$\operatorname{div}(\vec{v} - \vec{u}) = \underset{\substack{\uparrow \\ \text{incompressible}}}{\operatorname{div} v} - \underset{\substack{\uparrow \\ u = \text{const}}}{\operatorname{div} u} = 0$$

Then

$$\int \upsilon^2 dV = u^2 (V - V_0) + \int \mathrm{div}\left[(\varphi + \vec{u}\cdot\vec{r})(\vec{\upsilon} - \vec{u})\right] dV$$

Using Gauss theorem

$$\int \upsilon^2 dV = u^2 (V - V_0) + \int_{S+S_0} (\varphi + \vec{u}\cdot\vec{r})(\vec{\upsilon} - \vec{u})\, d\vec{S}$$

At the body surface S_0 $\vec{\upsilon}_n = \vec{u}_n \Rightarrow \int_{S_0} \ldots dS = 0$

Thus

$$\int \upsilon^2 dV = u^2 (V - V_0) + \int_{S} (\varphi + \vec{u}\cdot\vec{r})(\vec{\upsilon} - \vec{u})\, d\vec{S}$$

To calculate the integral over the remote surface we take $d\vec{S} = \vec{n} R_0^2 \, d\Omega$

$$\varphi = -\frac{\vec{A}\cdot\vec{n}}{R_0^2}, \quad \vec{\upsilon} = \frac{3(A\cdot\vec{n})\vec{n} - \vec{A}}{R_0^3}$$

Substituting we obtain

$$\int v^2 dV = u^2(V - V_0) + $$

$$+ \int \left[(\vec{u}\cdot\vec{n}) R_0 - \frac{A\cdot\vec{n}}{R_0^2} \right] \left\{ \frac{\left[3(\vec{A}\cdot\vec{n})\vec{n} - \vec{A}\right]}{R_0^3} - \vec{u} \right\} \vec{n}\, R_0^2 \, d\Omega =$$

$$= u^2(V - V_0) +$$

$$+ \frac{1}{R_0^3} \int \left[(\vec{u}\cdot\vec{n}) R_0^3 - (\vec{A}\cdot\vec{n}) \right] \left(2(\vec{A}\cdot\vec{n}) - (\vec{u}\cdot\vec{n}) R_0^3 \right) d\Omega =$$

$$= u^2(V - V_0) + \int \left[3(\vec{u}\cdot\vec{n})(\vec{A}\cdot\vec{n}) - (\vec{u}\cdot\vec{n})^2 R_0^3 \right] d\Omega -$$

$$- \frac{2}{R_0^3} \int (\vec{A}\cdot\vec{n})^2 \, d\Omega$$

The last term goes to zero as $R_0 \to \infty$

Then

$$\int v^2 dV = u^2 \left(\frac{4\pi}{3} R_0^3 - V_0 \right) + \int \left[3(\vec{u}\cdot\vec{n})(\vec{A}\cdot\vec{n}) - (\vec{u}\cdot\vec{n})^2 R_0^3 \right] d\Omega$$

The integration over Ω is equivalent to averaging over all directions of $\vec{n}$ and multiplying by 4π

$$\overline{(\vec{A}\cdot\vec{n})(\vec{u}\cdot\vec{n})} = A_i u_k \overline{n_i n_k} = A_i u_k \frac{\delta_{ik}}{3} = \frac{(\vec{A}\cdot\vec{u})}{3}$$

Then the terms with R_0^3 cancel out and

$$E = \int \rho \frac{v^2}{2} dV = \frac{\rho}{2}\left[4\pi(\vec{A}\cdot\vec{u}) - V_0 u^2\right]$$

Using $A_i = \alpha_{ik} u_k$ we obtain the energy

$$\underline{E = \frac{m_{ik} u_i u_k}{2}}$$

Here the <u>induced mass tensor</u> $m_{ik} = \rho(4\pi\alpha_{ik} - V_0\delta_{ik})$

For sphere $A = \frac{\vec{u}R^3}{2}$ $\Rightarrow$

$$E = \frac{\rho u^2}{2}\left(2\pi R^3 - \frac{4\pi}{3}R^3\right) = \rho \frac{2\pi}{3}R^3\frac{u^2}{2} =$$

$$= \rho \frac{V_0}{2}\frac{u^2}{2} \Rightarrow m_{ik} = \rho\frac{V_0}{2}\delta_{ik}$$

is half the mass of the displaced fluid

Knowing the energy E we can obtain an expression for the total momentum $\vec{P}$, of the fluid. The change in the energy of the body (= minus the change of the fluid energy dE) is the work done by force F on the path $u\,dt$ =>

$$dE = -\vec{F}\vec{u}\,dt \Rightarrow$$

$$F_i = -\frac{d}{dt} m_{ik} u_k = -\frac{d\vec{P}}{dt} \quad \text{with } P_i = m_{ik} u_k$$

Since in the potential flow $u = \text{const} \Rightarrow$ $F = 0 \Rightarrow$ all forces on the body are zero including the force parallel to $\vec{u}$ (<u>drag force</u>) and the force perpendicular to $\vec{u}$ (<u>lift force</u>)

<u>D'Alembert's paradox.</u> Flying is <u>impossible</u>

We can come to this result also by caclulating the momentum flux for the fluid.

$$\frac{\partial}{\partial t}\int \rho v_i \, dV = -\oint dS_k \, \Pi_{ik}, \quad \Pi_{ik} = p\delta_{ik} + \rho v_i v_k$$

Then $\frac{dP_i}{dt} = -\int_S (p\,\delta_{ik} + \rho\, v_i v_k)\, dS_k$

Since far away from the body $p = \text{const}$

$v \propto \frac{1}{r^3}$ then the integral vanishes.

If the fluid has a free surface a body moving parallel to it will experience a drag due to surface waves (wave drag).

Reversibility and D'Alembert's paradox

Euler and continuity equations are reversible if $v(r,t)$ is a solution with velocity at infinity equal to $\vec{u}$ then $w(r,t) = -v(r,-t)$ is also a solution with $w(\infty,t) = -u$ but with the same pressure and density field.

But the force on the body is integral of the pressure since it doesn't change with reversing of the flow it should be zero.

Suppose the body moves under the action of an external force $\vec{f}$. To find equation of motion of the body we should equate $\vec{f}$ to the time derivative of the total momentum of the system. The total momentum is the sum of the momentum of the body $M\vec{u}$ and the momentum $\vec{P}$ of the fluid.

$$M\frac{d\vec{u}}{dt}+\frac{d\vec{P}}{dt}=\vec{f}$$

$$\frac{d}{dt}\left(M\delta_{ik}+m_{ik}\right)u_k=f_i$$

Consider now an opposite situation when the fluid moves in an oscillating way while a small body is immersed into the fluid.

For example, a long sound wave propagates in a fluid. We want to relate the body velocity u to the fluid velocity v.

If the body moved with the same velocity $\vec{u} = \vec{v}$ then the force acting on it would be the same as the force acting on the fluid in its place $\rho V_0 \dot{\vec{v}}$. Relative motion gives the reaction force $\frac{d}{dt}(m_{ik}(v_k - u_k))$. Adding the forces gives the body acceleration

$$\frac{d(M u_i)}{dt} = \rho V_0 \dot{v}_i + \frac{d}{dt} m_{ik}(v_k - u_k)$$

Intergrating over time we obtain

$$\underline{(M \delta_{ik} + m_{ik}) u_k = (m_{ik} + \rho V_0 \delta_{ik}) v_k}$$

Here we put integration constant to zero because if $v = 0$ then $u = 0$.