Developed turbulence

As Reynolds number increases beyond the threshold of the first instability the new periodic flow is getting unstable with respect to another perturbation (usually with higher frequency and smaller scale). Every new instability brings about an extra degree of freedom. At very large Re, a sequence of instabilities produces turbulence as a superposition of different scales. The resulting flow is irregular both spatially and temporally.

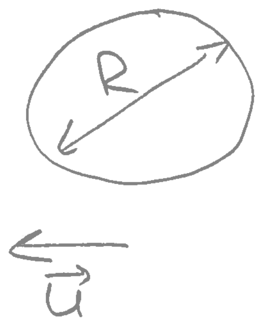
For rather small river $Re \sim \frac{u\,h}{\nu} \simeq 10^7$

$$u \sim 0.3\,\frac{m}{sec}, \quad L \sim 3\,m, \quad \nu \sim 10^{-6}\,\frac{m^2}{sec} \Rightarrow$$

Thus behaviour at very large Reynolds numbers is of great importance.

The empirical fact is that in this limit $\nu \to 0$, the rate of the energy dissipation stays finite.

To illustrate consider a body moving in a fluid at large Re. Let us calculate drag force by momentum balance.



During the time $\tau \sim \frac{R}{u}$ the body gets momentum $P \sim \rho R^3 u$ from the fluid.
The drag force $F \sim \frac{P}{\tau} \sim \rho R^2 u^2$
The energy dissipated per unit mass is

$$\varepsilon = \frac{F \cdot u}{\rho R^3} \sim \frac{\rho R^2 u^3}{\rho R^3} \sim \frac{u^3}{R}$$

To describe turbulence on more quantitative level we consider randomly forced Navier-Stokes equation:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \nu \vec{\nabla}^2 \vec{v} - \frac{\nabla p}{\rho} + \frac{\vec{f}(r,t)}{\rho}$$

with

$$\langle f_\alpha(t,r)\, f_\beta(t',r')\rangle = \delta(t-t')\,\chi_{\alpha\beta}(r-r')$$

Multiplying NS Eq. by $\vec{v}$ and integrating we obtain for the incompressible fluid

$$\frac{\partial}{\partial t}\int \frac{v^2}{2}\, d^d r = -\nu \int (\nabla_\alpha v_\beta)^2 d^d r + \int f\cdot\vec{v}\, d^d r$$

This equation states that the rate of change of fluid energy is equal to the energy injection rate $\int \vec{f}\cdot\vec{v}\, d^d r$ (work of the external force per unit time) minus the energy dissipation $\nu\int(\nabla_\alpha v_\beta)^2 d^d r$ due to the viscous friction.

In a stationary state, the mean energy is constant in time =>

$$\int \langle \nu (\nabla_\alpha v_\beta)^2 \rangle \, d^d r = \int \langle \vec{v} \cdot \vec{f} \rangle \, d^d r$$

We can define the mean dissipation rate ε:

$$\varepsilon \equiv \langle \nu (\nabla_\alpha v_\beta)^2 \rangle = \langle \vec{v} \cdot \vec{f} \rangle$$

The energy is injected (pumped) at large scales L, then for high Re there is a scale separation and dissipation takes place on much smaller viscous scales $\sim \lambda$. Energy is transmitted by the excitation of large eddies which break into a smaller and smaller eddies (without energy loss) until the viscous scale λ is reached, where energy is dissipated. This is the energy cascade picture (L.F. Richardson 1921).

L/λ is growing with the increase of Re.

A.N. Kolmogorov theory of developed turbulence (1941)

Scaling estimates

For a given scale l one can define scale dependent Reynolds number $Re_l \sim \frac{v_l l}{\nu}$.

Viscosity becomes important for $Re_\lambda \sim 1$

The region $\lambda \ll r \ll L$ is called the inertial range.

In this range natural assumption is that all the properties are independent on viscosity (more precisely do not change when $\nu \to 0$).

Consider energy dissipation rate ε. Although at the end dissipation is due to viscosity its order of magnitude can be determined by the large scale behaviour (dissipative anomaly).

The only parameters we have are scale l and velocity variation δu.

Since $[\varepsilon] = \frac{m^2}{sec^3} \Rightarrow$

$$\varepsilon \sim \frac{(\delta u)^3}{l}$$

This can be rewritten in a way that characteristic velocity variations over distances l

$$\underline{\delta v(l) \sim (\varepsilon l)^{1/3}} \quad \text{(Kolmogorov Obukhov law)}$$

Consider the scale dependent Reynolds number

$$Re_l \sim \frac{\delta v(l) l}{\nu} \sim \frac{(\varepsilon l)^{1/3} l}{\nu} \sim \frac{(\varepsilon L)^{1/3} L}{\nu} \left(\frac{l}{L}\right)^{4/3} \sim$$

$$\sim \frac{1}{\nu} (v^3)^{1/3} L \left(\frac{l}{L}\right)^{4/3} \sim Re \left(\frac{l}{L}\right)^{4/3}$$

From $Re_\lambda \sim 1$ we obtain the dissipative scale

$\lambda \sim L / Re^{3/4} \ll L$.

Note that the total number of degrees of freedom $N \sim \left(\frac{L}{\lambda}\right)^3 \sim Re^{9/4}$ which make numerical calculations extremely hard

Note that from the Kolmogorov Obukhov law follows that the separation between two point of fluid grows with time as

$$\delta^2(t) \propto t^3$$

L. F. Richardson (1926)

Exact formulation

Let us rewrite the NS equation as

$$\vec{v}(t+\Delta t) = \vec{v}(t) + \left[-(\vec{v}\cdot\vec{\nabla}\vec{v})\vec{v} + \nu\nabla^2\vec{v}\right]\Delta t + \int_t^{t+\Delta t} f(s)\,ds + O\left((\Delta t)^2\right)$$

Let us multiply this equation by the same at different point and average

$$\langle\vec{v}(t+\Delta t,x)\cdot\vec{v}(t+\Delta t,y)\rangle = \langle\vec{v}(t,x)\,\vec{v}(t,y)\rangle +$$

$$+\Big[-\langle\vec{v}(y)\cdot(\vec{v}(x)\cdot\vec{\nabla})\vec{v}(x)\rangle - \langle\vec{v}(x)\cdot(\vec{v}(y)\cdot\vec{\nabla})\vec{v}(y)\rangle +$$

$$+\nu\langle\vec{v}(y)\nabla^2\vec{v}(x)\rangle + \nu\,\vec{v}(x)\nabla^2\vec{v}(y) +$$

$$+ X_{dd}\left(\frac{x-y}{L}\right)\Big]\Delta t + O\left((\Delta t)^2\right)$$

Assuming homogeneity ($\langle\ldots\rangle = \Phi(x-y)$)

the second line terms may be rewritten as

$$\frac{1}{2}\vec{\nabla}_x\cdot\langle(\vec{v}(x)-\vec{v}(y))\,(v(x)-v(y))^2\rangle$$

Indeed $(\vec{v}(x)=\vec{v},\ v(y)=\vec{v}')$,

$$\langle (v_i - v_i')(v_k - v_k')^2 \rangle = \langle v_i v_k^2 \rangle - \langle v_i' v_k'^2 \rangle +$$

$$+ \langle v_i v_k'^2 \rangle - \langle v_i' v_k^2 \rangle +$$

$$+ 2\langle v_i' v_k v_k' \rangle - 2\langle v_i v_k v_k' \rangle$$

$\langle v_i v_k^2 \rangle - \langle v_i' v_k'^2 \rangle = 0$ because of homogeneity

The second line will get zero after $\frac{\partial}{\partial r_i}$ because $\operatorname{div} \vec{v} = 0$ and from the third line we get $-2 v_k' (v_i \cdot \nabla_i) v_k + 2 v_k (v_i' \nabla_i) v_k'$.

In the same way the third line on the long equation on the p. 212 is

$$-2\nu \langle \nabla_\alpha v_\beta(x) \nabla_\alpha v_\beta(y) \rangle$$

Then

$$\frac{\partial}{\partial t}\langle v(x)\,v(y)\rangle = \frac{1}{2}\vec{\nabla}_x\cdot\langle(\vec{v}(x)-\vec{v}(y))(v(x)-v(y))^2\rangle$$

$$-2\nu\langle\vec{\nabla}_\alpha v_\beta(x)\cdot\vec{\nabla}_\alpha v_\beta(y)\rangle + \chi_{\alpha\alpha}\left(\frac{x-y}{L}\right)$$

(von Kármán & Horwarth 1938)

In the stationary regime $\frac{\partial}{\partial t}\langle\ \rangle = 0 \Rightarrow$

$$-\frac{1}{4}\vec{\nabla}_x\cdot\langle(\vec{v}(x)-\vec{v}(y))(v(x)-v(y))^2\rangle + \nu\langle\nabla_\alpha v_\beta(x)\,\nabla_\alpha v_\beta(y)\rangle$$

$$= \frac{1}{2}\chi_{\alpha\alpha}\left(\frac{x-y}{L}\right)$$

Taking limit $x\to y$ at positive ν and assuming smooth behaviour $v(x)-v(y)\propto$ $\propto x-y$ at the distances smaller than the viscous scale the first term vanishes and $\varepsilon \equiv \nu\langle(\nabla_\alpha v_\beta)^2\rangle = \dfrac{\chi_{\alpha\alpha}(0)}{2}$

In the opposite regime $\nu \to 0, |x-y| > \lambda$

$$-\frac{1}{4}\vec{\nabla}_x \cdot \left\langle \left(\vec{v}(x)-\vec{v}(y)\right)\left(v(x)-v(y)\right)^2\right\rangle = \chi_{\alpha\alpha}\left(\frac{x-y}{L}\right)$$

For $|x-y| \ll L$ $\quad \chi\left(\frac{x-y}{L}\right) = \chi(0)$, then

using the last Eq. on p. 214 we obtain

$$-\frac{\partial}{\partial x_\alpha}\left\langle \left(v_\alpha(x)-v_\alpha(y)\right)\left(v_\beta(x)-v_\beta(y)\right)^2\right\rangle = \varepsilon$$

Assuming isotropy (rotational invariance) this implies

$$\left\langle \left(v_\alpha(x)-v_\alpha(y)\right)\left(v_\beta(x)-v_\beta(y)\right)\left(v_\gamma(x)-v_\gamma(y)\right)\right\rangle =$$

$$= -\frac{4\varepsilon}{d(d+2)}\left(\delta_{\alpha\beta}r_\gamma + \delta_{\alpha\gamma}r_\beta + \delta_{\beta\gamma}r_\alpha\right), \quad r = x-y$$

For the longitudinal 3 point structure function we obtain

$$S_3^{\parallel}(r) = \left\langle \left[\left(\vec{v}(0)-\vec{v}(\vec{r})\right)\cdot\frac{\vec{r}}{|r|}\right]^3\right\rangle = -\frac{12}{d(d+2)}\varepsilon r$$

In 3d it is $\quad S_3^{\parallel}(r) = -\frac{4}{5}\varepsilon r$

Kolmogorov (1941)

Intermittency

The energy spectrum

One can present the Kolmogorov Obukhov's law in a different (spectral) form. We replace scales l by corresponding wave numbers $k \sim \frac{1}{\lambda}$ of eddies.

Let $E(k)\,dk$ be the kinetic energy per unit mass in eddies with wave number between k and $k+dk$

$$E(k) = \frac{1}{2}\langle |v_k|^2 \rangle. \qquad E = \int E(k)\,dk$$

Dimensionally $[E(k)] = \frac{m^3}{sec^2}$ $\Rightarrow$

$$\underline{E(k) \propto \varepsilon^{2/3} k^{-5/3}}$$

v_l^2 gives the order of total energy in the eddies with scales less then l

Then $\int_{q\sim\frac{1}{l}}^{\infty} E(k)\,dk \sim \varepsilon^{2/3} q^{-2/3} \sim (\varepsilon l)^{2/3} \sim v_l^2$

in agreement with $v_l^3 \sim \varepsilon l$

In the same K41 paper Kolmogorov postulated <u>universality</u> of the cascade in the inertial range. This implies that

$$S_n^{\parallel}(r) = \langle [(\vec{v}(x)-\vec{v}(y))\cdot \frac{\vec{r}}{|r|}]^n \rangle = C_n \varepsilon^{n/3} |r|^{n/3}$$

with universal constants C_n

L.D. Landau questioned universality of $S_2(r)$, arguing that its instanteneous value might in principle be expressed as a function of the dissipation $\varepsilon(t)$ at the instant considered. After averaging important part of it should come from the pumping scale L and hence cannot be universal.

In general one can write

$$S_n^{\parallel} \propto r^{\xi_n}$$

Difference of ξ_n from the Kolmogorov values means appearence of another length scale (L) or absence of finite limit of the corresponding moments at $\nu \to 0$, $L \to \infty$

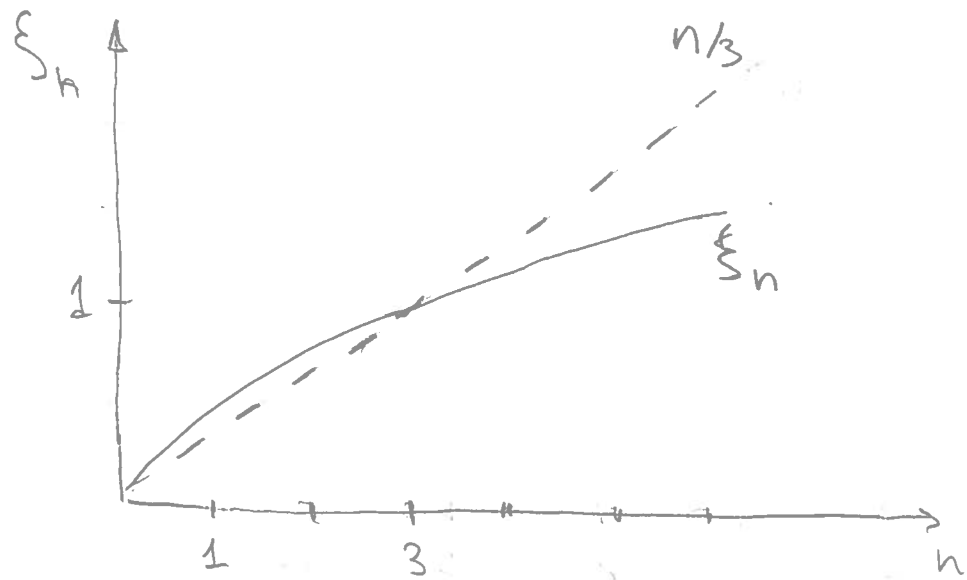
Experimentally

$\xi_2 = 0.7$, $\xi_3 = 1$, $\xi_4 = 1.28$, $\xi_5 = 1.53$, $\xi_6 = 1.77$, $\xi_7 = 2.01$

Kolmogorov values

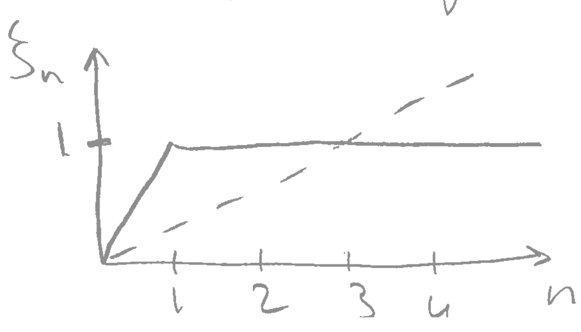
$\xi_2 = 0.67$, $\xi_3 = 1$, $\xi_4 = 1.33$, $\xi_5 = 1.67$, $\xi_6 = 2$, $\xi_7 = 2.33$

This difference is the main problem in turbulence for the third millenium.



The best studied (theoretically) case is 1d Burgers turbulence $\frac{\partial v}{\partial t} + v_x \frac{\partial v}{\partial x} = \nu \frac{\partial^2 v}{\partial x^2} + f$

where $\xi_n = 1$ for $n \geq 1$, $\xi_n = n$ for $0 \leq n \leq 1$



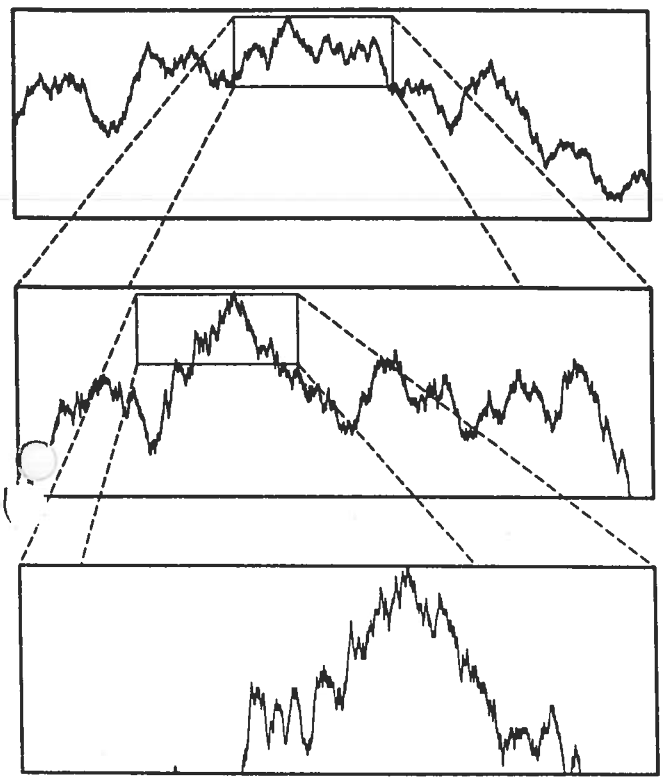


Fig. 8.1. A portion of the graph of the Brownian motion curve, enlarged twice, illustrating its self-similarity.

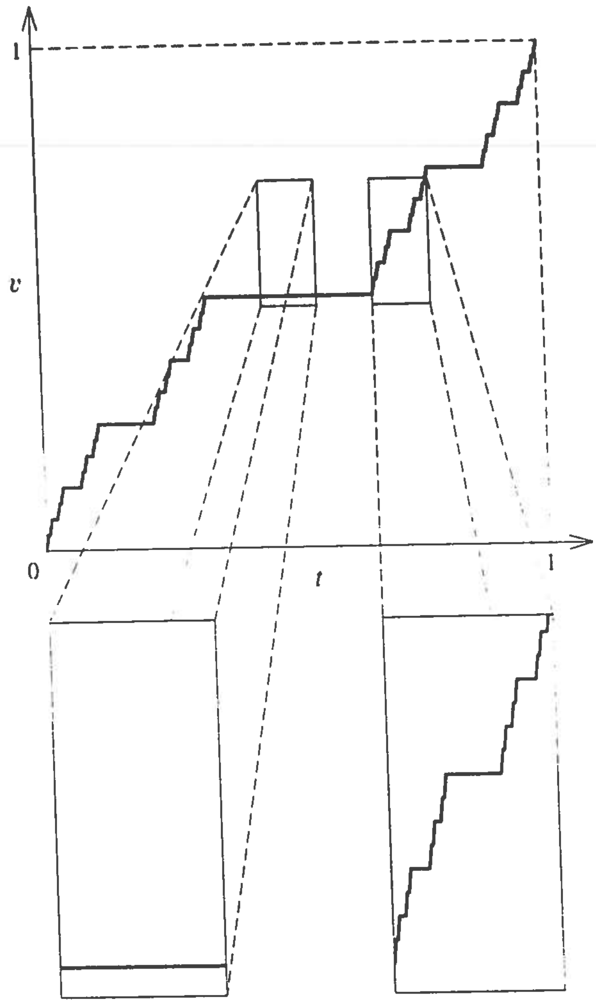


Fig. 8.2. The Devil's staircase: an intermittent function.

8.2 Self-similar and intermittent random functions

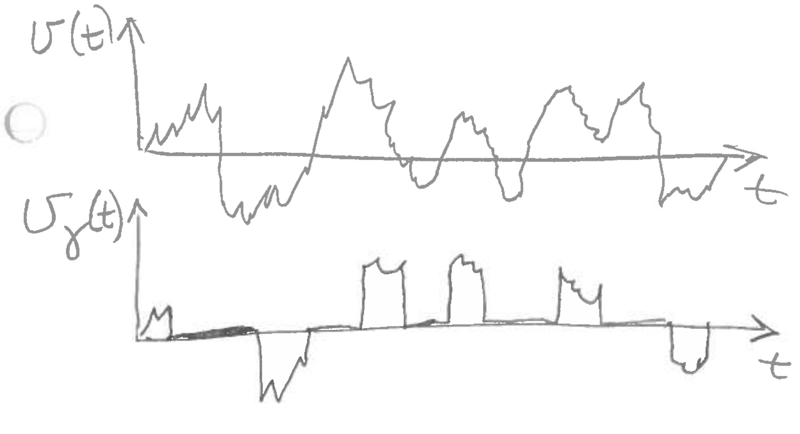
A central assumption of the K41 theory theory is the *self-similarity* of the random velocity field at inertial-range scales. As we shall see this symmetry may well be broken. The meaning of the concept of self-si[illegible]ity as applied to a random function is illustrated in Fig. 8.1 in w[illegible] a sample of a self-similar random function $v(t)$, here the Brownian m[illegible]n function, is shown with two successive enlargements. [illegible]

[1] The Devil's staircase gives the fraction of the mass of the Cantor set in the interval $[0, t]$. It is constructed recursively. One starts with a uniform distribution in the interval $[0, 1]$ of unit total mass, removes the middle third and redistributes the removed mass evenly among the remaining intervals. The process is repeated indefinitely.

The <u>anamolous scaling</u> observed experimentally means, that the distribution of $v(t,x)$ is non Gaussian. The discrepency may be measured by the skewness $S_3/S_2^{3/2}$ or the flatness S_4/S_2^2 which grow with $l \to 0$.

Intuitively one can imagine intermittency as if only a part of the fluid modes participates in the turbulent cascade, with the proportion of active modes decreasing with diminishing scale.

As an example, consider signal



stationary random signal

the same signal where $(1-\gamma)$ of it is cut.

Then $\langle v_\gamma^2 \rangle = \gamma \langle v^2 \rangle$, $\langle v_\gamma^4 \rangle = \gamma \langle v^4 \rangle \Rightarrow$

$$\frac{\langle v_\gamma^4 \rangle}{(\langle v_\gamma^2 \rangle)^2} = \frac{1}{\gamma} \frac{\langle v^4 \rangle}{(\langle v^2 \rangle)^2}$$

In this example moments of v are determined by rare (: probability δ) large fluctuations. May be something like this happens in real turbulence. There are multifractal models of that type. Such models, although interesting phenomenologically do not follow from the Navier Stokes equation. Thus they do not really explain the breaking of scaling in realistic flows.

Note, that without doubt (neither theoretical nor experimental) there is no need to modify the Navier Stokes equation in order to get turbulence. Scaling with Re works both for laminar and turbulent flow. It is our unability to solve it that prevents us from understing it.

State-of-the-art experiments and computations are certainly a prerequisite for progress in turbulence. However, it is a long way from measuring and seeing everything to understanding. Indeed, turbulent flow has been observed carefully for five centuries, measured for a century and simulated for a quarter of a century. Fig. 9.1, a cartoon of the state-of-the-art drawn in 1977 by Philippe Delache, remains largely valid today. Such long time scales are most unusual in physics but are occasionally encountered in mathematics. Is it by accident that the deepest insight into turbulence came from Andrei Nikolaevich Kolmogorov, a mathematician with a keen interest in the real world?

last paragraph of the book U. Frisch "Turbulence"

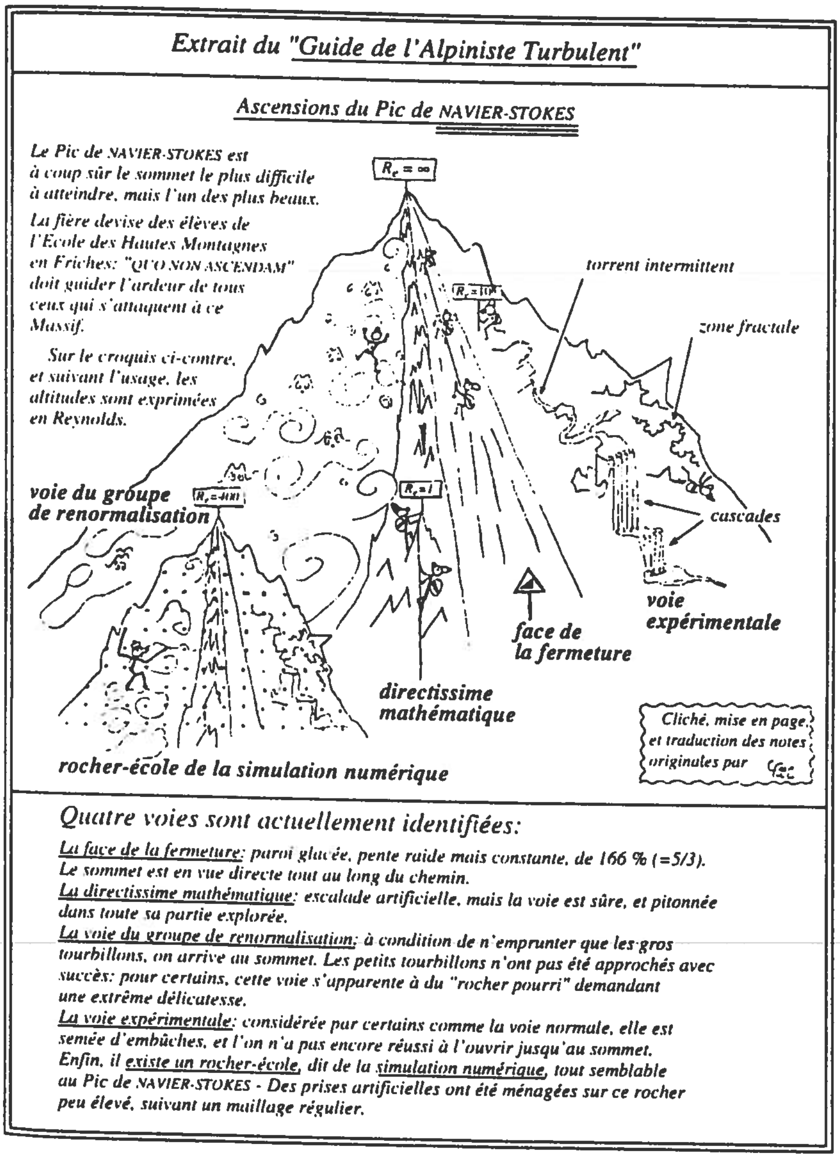


Fig. 9.1. Cartoon drawn in 1977 by the astronomer Philippe Delache, a penetrating observer of the turbulence community. He was the author's friend and died prematurely in 1994. The figure shows the 'Navier–Stokes peak' and four explored faces: experimentation, closure, mathematics and renormalization. It also shows a reduced model, the rock-climbing school of numerical simulation.

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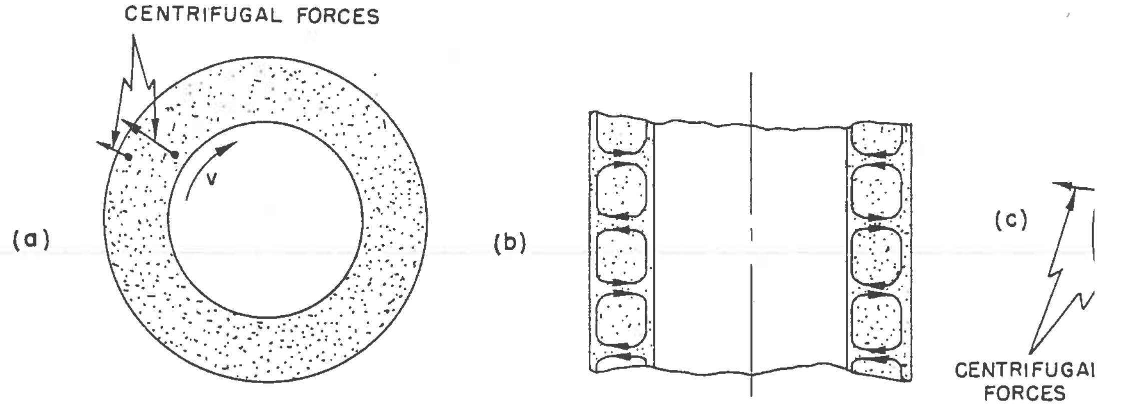

Fig. 41–9. Why the flow breaks up into bands.

knows why! There's a challenge. A simple number like 1/3, and no explanation. In fact, the whole mechanism of the wave formation is not very well understood; yet it is steady laminar flow.

If we now start rotating the outer cylinder also—but in the opposite direction—the flow pattern starts to break up. We get wavy regions alternating with apparently quiet regions, as sketched in Fig. 41–8(d), making a spiral pattern. In these "quiet" regions, however, we can see that the flow is really quite irregular; it is, in fact completely turbulent. The wavy regions also begin to show irregular turbulent flow. If the cylinders are rotated still more rapidly, the whole flow becomes chaotically turbulent.

In this simple experiment we see many interesting regimes of flow which are quite different, and yet which are all contained in our simple equation for various values of the one parameter $\mathcal{R}$. With our rotating cylinders, we can see many of the effects which occur in the flow past a cylinder: first, there is a steady flow; second, a flow sets in which varies in time but in a regular, smooth way; finally, the flow becomes completely irregular. You have all seen the same effects in the column of smoke rising from a cigarette in quiet air. There is a smooth steady column followed by a series of twistings as the stream of smoke begins to break up, ending finally in an irregular churning cloud of smoke.

The main lesson to be learned from all of this is that a tremendous variety of behavior is hidden in the simple set of equations in (41.23). All the solutions are for the same equations, only with different values of $\mathcal{R}$. We have no reason to think that there are any terms missing from these equations. The only difficulty is that we do not have the mathematical power today to analyze them except for very small Reynolds numbers—that is, in the completely viscous case. That we have written an equation does not remove from the flow of fluids its charm or mystery or its surprise.

NS. Eq

If such variety is possible in a simple equation with only one parameter, how much more is possible with more complex equations! Perhaps the fundamental equation that describes the swirling nebulae and the condensing, revolving, and exploding stars and galaxies is just a simple equation for the hydrodynamic behavior of nearly pure hydrogen gas. Often, people in some unjustified fear of physics say you can't write an equation for life. Well, perhaps we can. As a matter of fact, we very possibly already have the equation to a sufficient approximation when we write the equation of quantum mechanics:

$$H\psi = -\frac{\hbar}{i}\frac{\partial\psi}{\partial t}.$$

We have just seen that the complexities of things can so easily and dramatically escape the simplicity of the equations which describe them. Unaware of the scope of simple equations, man has often concluded that nothing short of God, not mere equations, is required to explain the complexities of the world.

We have written the equations of water flow. From experiment, we find a set of concepts and approximations to use to discuss the solution—vortex streets, turbulent wakes, boundary layers. When we have similar equations in a less familiar situation, and one for which we cannot yet experiment, we try to solve the equations in a primitive, halting, and confused way to try to determine what new qualitative features may come out, or what new qualitative forms are a consequence of the equations. Our equations for the sun, for example, as a ball of hydrogen gas, describe a sun without sunspots, without the rice-grain structure of the surface, without prominences, without coronas. Yet, all of these are really in the equations; we just haven't found the way to get them out.

There are those who are going to be disappointed when no life is found on other planets. Not I—I want to be reminded and delighted and surprised once again, through interplanetary exploration, with the infinite variety and novelty of phenomena that can be generated from such simple principles. The test of science is its ability to predict. Had you never visited the earth, could you predict the thunderstorms, the volcanos, the ocean waves, the auroras, and the colorful sunset? A salutary lesson it will be when we learn of all that goes on on each of those dead planets—those eight or ten balls, each agglomerated from the same dust cloud and each obeying exactly the same laws of physics.

The next great era of awakening of human intellect may well produce a method of understanding the *qualitative* content of equations. Today we cannot. Today we cannot see that the water flow equations contain such things as the barber pole structure of turbulence that one sees between rotating cylinders. Today we cannot see whether Schrödinger's equation contains frogs, musical composers, or morality —or whether it does not. We cannot say whether something beyond it like God is needed, or not. And so we can all hold strong opinions either way.

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