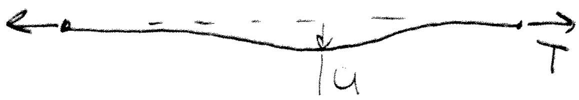
Elastic waves

Elastic string under tension T

Its elastic energy is

$$E_{el} = \int \frac{T}{2} \left(\frac{\partial u}{\partial x}\right)^2 dx$$



Kinetic energy is

$$E_k = \int \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx$$

ρ is linear mass density

Lagrangean

$$L = \int\left[\frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{T}{2}\left(\frac{\partial u}{\partial x}\right)^2\right] dx\, dt$$

varying we obtain equation of motion

$$\rho \frac{\partial^2 u}{\partial t^2} - T \frac{\partial^2 u}{\partial x^2} = 0$$

This is the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

With the sound velocity $c = \sqrt{\frac{T}{\rho}}$

It has general solution in the form

$$u = f(x - ct) + g(x + ct),$$

where f and g are arbitrary functions.

We could also get the same wave equation from the Newton's second law,

$$\vec{F} = m\vec{a},$$

$$T \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2}$$

Elastic waves in isotropic medium

Equation of motion:

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ik}}{\partial x_k}$$

In vector form

$$\rho \ddot{u} = \frac{E}{2(1+\sigma)} \nabla^2 \vec{u} + \frac{E}{2(1+\sigma)(1-2\sigma)} \text{grad div } u$$

or

$$\rho \ddot{u} = \mu \nabla^2 u + (\mu + \lambda) \text{grad div } \vec{u}$$

One has two kind of waves $\vec{u} = \vec{u}_e + \vec{u}_t$

longitudinal $(\text{rot } \vec{u}_e = 0)$ and

transverse $(\text{div } \vec{u}_t = 0)$

For longitudinal waves we can write

$$\nabla^2 u = \text{grad div}\,\vec{u} - \underbrace{\text{rot rot}\, u}_{\overset{\shortparallel}{0}} = \text{grad div}\, u_e \Rightarrow$$

$$\rho \ddot{u}_e = (2\mu + \lambda)\nabla^2 u_e$$

$$\ddot{\vec{u}}_e - c_e^2 \nabla^2 \vec{u}_e = 0$$

with $c_e = \sqrt{\dfrac{2\mu + \lambda}{\rho}}$

For transverse waves $\text{div}\, u = 0$ and

$$\rho \ddot{\vec{u}}_t = \mu \nabla^2 \vec{u}_t \Rightarrow$$

$$\ddot{\vec{u}}_t - c_t^2 \nabla^2 \vec{u}_t = 0$$

with $c_t = \sqrt{\dfrac{\mu}{\rho}}$

Because $K = \lambda + \frac{2}{3}\mu > 0$ we can rewrite

$$c_e = \sqrt{\frac{\frac{4}{3}\mu + K}{\rho}} > \sqrt{\frac{4}{3}}\, c_t$$

Since usually $\lambda > 0 \Leftrightarrow K > \frac{2}{3}\mu \Rightarrow c_e > \sqrt{2}\, c_t$

In transverse wave displacement is perpendicular in a plane to direction of propagation. It does not involve any change in volume. In longitudinal wave there are compressions and expansions in the body. These longitudinal waves exist also in liquids.

For monochromatic plane wave

$$\vec{u} = \mathrm{Re}\left(\vec{A}_k e^{i(\vec{k}x - \omega t)}\right)$$

dispersion

For longitudinal $\vec{A}_k \parallel \vec{k}$, $\omega_e = c_e k$

For transverse $\vec{A}_k \perp \vec{k}$, $\omega_t = c_t k$

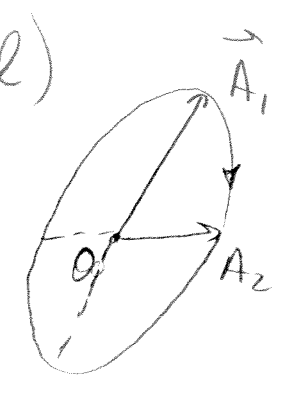
Polarization

For transverse wave one can write

$\vec{A} = \vec{A}_1 + i\vec{A}_2$ ($\vec{A}_1, \vec{A}_2$ are real)

Then at given point (e.g. $x=0$)

$u(t) = \vec{A}_1 \cos\omega t + \vec{A}_2 \sin\omega t$ - ellipse



Linear polarization $\vec{A}_1 \parallel \vec{A}_2$

Circular polarization $\vec{A}_1 \perp \vec{A}_2$ $|A_1| = |A_2|$

Elastic waves in crystals

Equation of motion

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ik}}{\partial x_k}$$, but now $\sigma_{ik} = \lambda_{iklm} u_{lm}$ =>

$$\Rightarrow \rho \frac{\partial^2 u_i}{\partial t^2} = \lambda_{iklm} \frac{\partial^2 u_m}{\partial x_k \partial x_l}$$

Searching $\vec{u}(r,t) = \vec{A} e^{i(\vec{k}\vec{r}-\omega t)}$ we get

$$\rho \omega^2 A_i = \lambda_{iklm} K_k K_l A_m$$ or

$$\left(\rho \omega^2 \delta_{im} - \lambda_{iklm} K_k K_l\right) A_m = 0$$

Since it is system of 3 homogeneous linear Eqs. it has solution only if

$$\mathrm{Det}\left| \lambda_{iklm} K_k K_l - \rho \omega^2 \delta_{lm} \right| = 0$$

It gives dispersion relation between ω and k

In general it is cubic equation for ω^2 that has 3 branches.

Example: cubic crystal

$\lambda_{xxxx} = C_{11}$, $\lambda_{xxyy} = C_{12}$, $\lambda_{xyxy} = C_{44}$

1) $\vec{k} = (k, 0, 0)$

$$\lambda_{iklm} k_k k_l = \lambda_{ixxm} k^2 = \begin{pmatrix} \lambda_{xxxx} k^2 & 0 & 0 \\ 0 & \lambda_{yxxy} k^2 & 0 \\ 0 & 0 & \lambda_{zxxz} k^2 \end{pmatrix}$$

$$\mathrm{Det} \begin{pmatrix} C_{11} k^2 - \omega^2 & 0 & 0 \\ 0 & C_{44} k^2 - \omega^2 & 0 \\ 0 & 0 & C_{44} k^2 - \omega^2 \end{pmatrix} = 0 \Rightarrow$$

One longitudinal wave $\omega_l^2 = \frac{C_{11}}{\rho} k^2$ and two transverse waves with $\omega_t^2 = \frac{C_{44}}{\rho} k^2$

2) $\vec{k} = \frac{1}{\sqrt{3}}(k, k, k)$

$$\mathrm{Det} \begin{pmatrix} \frac{1}{3}(C_{11} + 2C_{44}) k^2 - \rho\omega^2 & \frac{1}{3}(C_{12} + C_{44}) k^2 & \frac{1}{3}(C_{12} + C_{44}) k^2 \\ \frac{1}{3}(C_{12} + C_{44}) k^2 & \frac{1}{3}(C_{11} + 2C_{44}) k^2 - \rho\omega^2 & \frac{1}{3}(C_{12} + C_{44}) k^2 \\ \frac{1}{3}(C_{12} + C_{44}) k^2 & \frac{1}{3}(C_{12} + C_{44}) k^2 & \frac{1}{3}(C_{11} + 2C_{44}) k^2 - \rho\omega^2 \end{pmatrix}$$

It has the following form

$$\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix} = 0 \Rightarrow$$

$$a^3 + 2b^3 - 2ab^2 - b^2 a = 0 \Rightarrow$$

$$(a^2 - b^2)a + 2b^2(b-a) = 0 \Rightarrow$$

$$(a-b)\left[(a+b)a - 2b^2\right] = 0 \Rightarrow$$

$$(a-b)(a-b)(a+2b) = 0 \Rightarrow$$

$$a = b \quad , \quad a = -2b$$

Taking $a = \frac{1}{3}(C_{11} + 2C_{44})k^2 - \rho\omega^2$

$$b = \frac{1}{3}(C_{12} + C_{44})k^2$$

we obtain

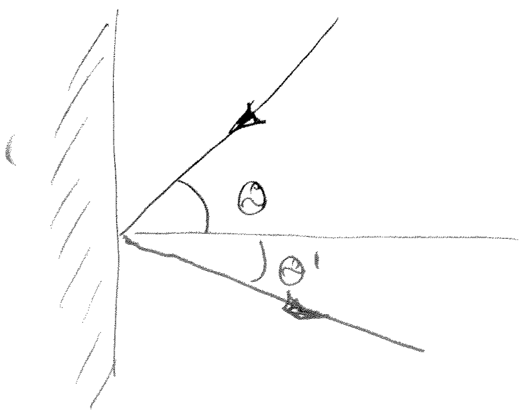
$$\omega_e^2 = \frac{1}{3}(C_{11} + 2C_{12} + 4C_{44})k^2$$

$$\omega_t^2 = \frac{1}{3}(C_{11} - C_{12} + C_{44})k^2$$

Reflection at a free surface

In general, nature of wave is changed when it is reflected or refracted. If a purely longitudinal or purely transverse wave is incident, the result is a mixed wave containing both transverse and longitudinal parts

The relation between the directions of incident and refected waves can be obtained as in optics from the constancy of phase of all the waves at the surface. Thus



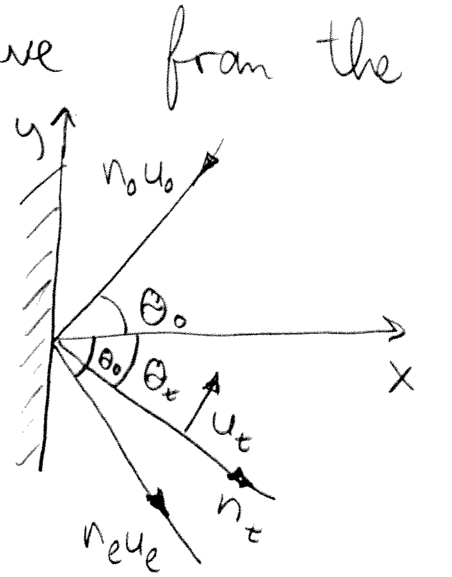
$$\omega = \omega' , \quad K_{\parallel} = K'_{\parallel} \Rightarrow$$

$$K \sin\Theta = K' \sin\Theta'$$

$$K = \frac{\omega}{c} , \quad K' = \frac{\omega}{c'} \Rightarrow$$

$$\frac{\sin\Theta}{\sin\Theta'} = \frac{c}{c'} = n$$

Consider, for example, reflection of the longitudinal wave from the surface of a body



There is longitudinal and transverse reflected waves with $\sin\theta_t = \frac{c_t}{c_e}\sin\theta_0 = \frac{\sin\theta_0}{n}$

From the symmetry we see, that $\vec{u}_t$ is in the plane of incidence. Total displacement (upto $e^{-i\omega t}$)

$$\vec{u} = A_0 n_0 e^{i\vec{k}_0\vec{r}} + A_e n_e e^{i\vec{k}_e\vec{r}} + A_t[\vec{a}\times\vec{n}_t]\, e^{i\vec{k}_t\cdot\vec{r}}$$

where $\vec{a}$ is unit vector $\parallel z$

Strain tensor at the boundary is (upto a phase factor)

$$u_{xx} = i k_0(A_0+A_e)\cos^2\theta_0 + iA_t k_t \cos\theta_t \sin\theta_t$$

$$u_{ee} = i k_0 (A_0 + A_e)$$

$$u_{xy} = i k_0 (A_0 - A_e)\sin\theta_0\cos\theta_0 + \frac{1}{2} i A_t k_t(\cos^2\theta_t - \sin^2\theta_t)$$

At the border we should have

$\sigma_{xx} = \sigma_{yx} = 0$

We can express $\sigma_{ik} = 2\rho c_t u_{ik} + \rho (c_e^2 - 2c_t^2) u_{ee} \delta_{ik}$

Then we have two equations for A_o, A_e, A_t that give

$$A_e = A_o \frac{\sin 2\theta_t \sin 2\theta_o - n^2 \cos^2 2\theta_t}{\sin 2\theta_t \sin 2\theta_o + n^2 \cos^2 2\theta_t} \qquad n = \frac{c_e}{c_t}$$

$$A_t = -A_o \frac{2n \sin 2\theta_o \sin 2\theta_t}{\sin 2\theta_t \sin 2\theta_o + n^2 \cos^2 2\theta_t}$$

For $\theta_o = 0$ we have $A_e = -A_o$, $A_t = 0 \Rightarrow$ the wave is reflected as completely longitudinal wave.