# Electrodynamics

Problem Sets

ETH/Uni Zurich, FS14

Prof. N. Beisert

# Contents

## 1.1. Vector calculus

In this problem we recall a number of standard identities of vector calculus which we will frequently use in electrodynamics.

*Definitions/conventions:* We commonly write the well-known vectorial differentiation operators grad, div, rot using the vector $\vec{\nabla}$ of partial derivatives $\nabla_i := \partial/\partial x_i$ as

$$\operatorname{grad} F := \vec{\nabla} F\,, \qquad \operatorname{div} \vec{A} := \vec{\nabla}\cdot\vec{A}\,, \qquad \operatorname{rot} \vec{A} := \vec{\nabla}\times\vec{A}\,. \tag{1.1}$$

The components of a three-dimensional vector product $\vec{a}\times\vec{b}$ are given by

$$(\vec{a}\times\vec{b})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} a_j b_k\,. \tag{1.2}$$

Here, $\varepsilon_{ijk}$ is the totally anti-symmetric tensor in $\mathbb{R}^3$ with $\varepsilon_{123} = +1$.

**a)** Show that

$$\sum_{i=1}^{3} \varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \qquad \text{and} \qquad \sum_{i,j=1}^{3} \tfrac{1}{2}\varepsilon_{ijk}\varepsilon_{ijl} = \delta_{kl}\,. \tag{1.3}$$

**b)** Show the following identities for arbitrary vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$:

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})\,, \tag{1.4}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}\,, \tag{1.5}$$

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})\,. \tag{1.6}$$

**c)** Prove the following identities for arbitrary scalar fields $F$ and vector fields $\vec{A}$, $\vec{B}$:

$$\vec{\nabla}\times(\vec{\nabla}F) = 0\,, \tag{1.7}$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0\,, \tag{1.8}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \Delta\vec{A}\,, \tag{1.9}$$

$$\vec{\nabla}\cdot(F\vec{A}) = (\vec{\nabla}F)\cdot\vec{A} + F\vec{\nabla}\cdot\vec{A}\,, \tag{1.10}$$

$$\vec{\nabla}\times(F\vec{A}) = (\vec{\nabla}F)\times\vec{A} + F\vec{\nabla}\times\vec{A}\,, \tag{1.11}$$

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} + \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A})\,, \tag{1.12}$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})\,. \tag{1.13}$$

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## 1.2. Gauß's theorem

Consider the following vector fields $\vec{A}_i$ in two dimensions

$$\vec{A}_1 = \big(3xy(y-x), x^2(3y-x)\big)\,, \tag{1.14}$$

$$\vec{A}_2 = \big(x^2(3y-x), 3xy(x-y)\big)\,, \tag{1.15}$$

$$\vec{A}_3 = \big(x/(x^2+y^2), y/(x^2+y^2)\big) = \vec{x}/\vec{x}^{\,2}\,. \tag{1.16}$$

**a)** Compute the flux of $\vec{A}_i$ through the boundary of the square $Q$ with corners $\vec{x} = (\pm 1, \pm' 1)$

$$I_i = \oint_{\partial Q} dx\, \vec{n}\cdot\vec{A}_i\,. \tag{1.17}$$

**b)** Compute the divergence of $\vec{A}_i$ and its integral over the area of this square $Q$

$$I'_i = \int_Q d^2x\, \vec{\nabla}\cdot\vec{A}_i\,. \tag{1.18}$$

## 1.3. Stokes' theorem

Consider the vector field

$$\vec{A} = (x^2y, x^3 + 2xy^2, xyz)\,. \tag{1.19}$$

**a)** Compute the integral along the circle $S$ around the origin in the $xy$-plane with radius $R$

$$I = \oint_S d\vec{x}\cdot\vec{A}\,. \tag{1.20}$$

**b)** Compute the rotation $\vec{B}$ of the vector field $\vec{A}$

$$\vec{B} = \vec{\nabla}\times\vec{A}\,. \tag{1.21}$$

**c)** Compute the flux of the rotation $\vec{B}$ through the disk $S$ whose boundary is $S$, $\partial D = S$

$$I' = \int_D d^2x\, \vec{n}\cdot\vec{B}\,. \tag{1.22}$$

## 2.1. Potential and electric field strength

Four point charges are placed at the corners $(a, 0)$, $(a, a)$, $(0, a)$, $(0, 0)$ of a square. Find the potential and the electric field strength in the plane of this square. Sketch the field lines and equipotential lines of the following charge distributions:

**a)** $+q, +q, +q, +q$;

**b)** $-q, +q, -q, +q$;

**c)** $+q, +q, -q, -q$.

*Hint:* Using Mathematica, the commands `ContourPlot` and `StreamPlot` might come in handy.

## 2.2. Stable equilibrium

Two balls, each with charge $+q$, are placed on an isolator plate within the $z = 0$ plane where they can move freely without friction. Under the plate, a third ball with charge $-2q$ is fixed at $\vec{r} = (0, 0, -b)$. Treat the balls as point charges, and find stable positions for the two balls on the plate.

## 2.3. Energy stored in a parallel plate capacitor

Two plates with charges $+Q$ and $-Q$ and area $A$ are placed parallelly to each other at a (small) distance $d$. Find the energy stored in the electric field between them.

## 2.4. Electric field strength in a hollow sphere

A charged ball with homogeneous charge density $\rho$ and radius $R_{\mathrm{A}}$ contains a spherical cavity with radius $R_{\mathrm{I}}$ that is shifted from the centre by the vector $\vec{a}$ with $|\vec{a}| < R_{\mathrm{A}} - R_{\mathrm{I}}$. Calculate the electric field strength within the cavity.

*Hint:* Use Gauß's theorem and the superposition principle to calculate the field strength.

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## 2.5. Delta function

The delta function is often used to describe point charge densities. It is defined through the following property when integrated over a smooth test function $f$ with compact support

$$\int_{-\infty}^{\infty} dx\, f(x)\, \delta(x-a) = f(a)\,. \tag{2.1}$$

**a)** Show that it can be written as the limit $\lim_{\epsilon\to 0} g_\epsilon(x) = \delta(x)$ of following function

$$g_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}}\, e^{-x^2/(2\epsilon)}\,. \tag{2.2}$$

**b)** Show also that $\lim_{\epsilon\to 0} h_\epsilon(x) = \delta(x)$

$$h_\epsilon(x) = \frac{1}{2\pi i}\left(\frac{1}{x-i\epsilon} - \frac{1}{x+i\epsilon}\right). \tag{2.3}$$

**c)** Show, that the derivative of the delta function satisfies

$$\int_{-\infty}^{\infty} dx\, f(x)\, \delta'(x-a) = -f'(a)\,. \tag{2.4}$$

## 3.1. Imaginary dipoles

Consider two point charges $q$ and $q'$ at a distance $d$ from each other, and a plane perpendicular to the line through $q$ and $q'$ in a distance $\alpha d$ from $q$.

**a)** Show that, in order for the plane to be at constant potential, one must have $q' = -q$ and $\alpha = 1/2$.

*Hint*: Look at the potential at large distance first.

Now consider a point charge at the (cartesian) coordinates $(a, b, 0)$ in the empty region of a space filled with a grounded conductor except for positive $x$ and $y$.

**b)** Argue that introducing two mirror charges—with respect to the planes $x = 0$ and $y = 0$, respectively—is not sufficient to reproduce the boundary conditions of the conductor.

**c)** Exploiting the symmetry of the problem, introduce one more appropriate virtual charge and show explicitly that this makes the potential on the planes constant.

Finally, consider a point charge in the empty region of a space filled with a grounded conductor except for the region of the angle $0 \leq \varphi \leq \pi/n$ ($n$ integer).

**d)** Determine the distribution of imaginary charges that reproduces the electric field of this charge in the empty region of space. What is the value of the electric field on the line where the two faces of the conductor meet?

## 3.2. Conducting sphere in an external electric field

A conducting sphere, bearing total charge $Q$, is introduced into a homogeneous electric field $\vec{E} = E\vec{e}_z$. How does the electric field change due to the presence of the sphere? How is the charge distributed on the surface of the sphere?

*Hint*: Motivate the following ansatz in spherical coordinates

$$\Phi(r, \vartheta, \varphi) = f_0(r) + f_1(r) \cos\vartheta\,, \tag{3.1}$$

and solve the Poisson equation $\Delta\Phi = 0$ using the Laplace operator

$$\Delta\Phi(r, \vartheta, \varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Phi}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\Phi}{\partial\varphi^2}\,. \tag{3.2}$$

A set of boundary conditions that completely fixes the solution is:

- at distances far from the sphere, only the homogeneous field shall remain;
- the surface of the conducting sphere has to be an equipotential surface;
- the electric field has to satisfy Gauß' theorem.

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## 3.3. Spherical cavity

Consider a spherical cavity with radius $R$ and let the potential on its boundary be specified by an arbitrary function $U(\vartheta, \varphi)$.

**a)** Show that one can write the potential as

$$\Phi(\vec{x}) = \int \frac{R(R^2 - r^2)\, U(\vartheta', \varphi')}{4\pi(r^2 + R^2 - 2rR\cos\gamma)^{3/2}} \sin\vartheta'\, d\vartheta'\, d\varphi', \tag{3.3}$$

where $\gamma$ is the angle between $\vec{x}$ and $\vec{x}'$. Determine $\cos\gamma$ in terms of the variables $\vartheta$, $\varphi$, $\vartheta'$ and $\varphi'$.

*Hint:* Use the Green function obtained with the method of imaginary charges and re-express vectors therein with spherical coordinates.

**b)** Write down the most general solution of the Laplace equation in terms of spherical harmonics $Y_{l,m}$, then use orthonormality relations to determine the coefficients for the given boundary condition.

**c)** Write down the explicit potential $\Phi(\vec{x})$ inside the sphere for the boundary condition

$$U(\vartheta, \varphi) = U_0 \cos\vartheta\,. \tag{3.4}$$

## 3.4. Capacities

A simple capacitor consists of two isolated conductors that are oppositely (but equally in size) charged with $Q_1 = +Q$ and $Q_2 = -Q$. In general both conductors will have different electrical potential and $U = \Phi_1 - \Phi_2$ denotes the potential difference. A characteristic quantity of the capacitor is the capacity $C$ defined by

$$C = \frac{|Q|}{|U|}\,. \tag{3.5}$$

Calculate the capacities for the following cases:

**a)** two big parallel planar surfaces with area $A$ and small distance $d$;

**b)** two concentrical, conducting spheres with radii $a$ and $b$ ($b > a$);

**c)** two coaxial, conducting cylindrical surfaces of length $L$ and radii $a$, $b$ ($L \gg b > a$).

## 4.1. Spherical multipole moments of a cube

Positive and negative point charges $\pm q$ are located on the corners of a cube with side length $a$. Charges on neighboring corners have opposite signs. The point of origin is the center of the cube, and the edges are aligned with the $x, y, z$-axes. Let the charge at $x, y, z > 0$ be positive.

**a)** Determine the positions of the charges in cartesian and spherical coordinates.

**b)** Determine the charge density in cartesian coordinates and, thereafter, in spherical coordinates, using

$$\delta^3(\vec{x} - \vec{x}_0) = \frac{1}{r^2 \sin\vartheta}\,\delta(r - r_0)\,\delta(\vartheta - \vartheta_0)\,\delta(\varphi - \varphi_0)\,, \tag{4.1}$$

as well as $\sin\vartheta = \sin(\pi - \vartheta)$ and $\cos(\pi - \vartheta) = -\cos\vartheta$.

**c)** Calculate the spherical dipole, quadrupole and octupole moments of this charge configuration, using

$$q_{l,m} = \int d^3r\,\rho(\vec{r})\,r^l\,Y^*_{l,m}(\vartheta, \varphi)\,, \qquad m = -l, -(l-1), \ldots, +(l-1), +l\,. \tag{4.2}$$

The required spherical harmonics are:

$$\begin{aligned}
Y_{00} &= 1, & Y_{11} &= -\sqrt{\frac{3}{2}}\,\sin\vartheta\,e^{i\varphi},\\
Y_{10} &= \sqrt{3}\,\cos\vartheta, & Y_{22} &= \sqrt{\frac{15}{8}}\,\sin^2\vartheta\,e^{2i\varphi},\\
Y_{21} &= -\sqrt{\frac{15}{2}}\,\cos\vartheta\,\sin\vartheta\,e^{i\varphi}, & Y_{20} &= \sqrt{\frac{5}{4}}\left(3\cos^2\vartheta - 1\right),\\
Y_{33} &= -\sqrt{\frac{35}{16}}\,\sin^3\vartheta\,e^{3i\varphi}, & Y_{32} &= \sqrt{\frac{105}{8}}\,\cos\vartheta\,\sin^2\vartheta\,e^{2i\varphi},\\
Y_{31} &= -\sqrt{\frac{21}{16}}\left(5\cos^2\vartheta - 1\right)\sin\vartheta\,e^{i\varphi}, & Y_{30} &= \sqrt{\frac{7}{4}}\left(5\cos^3\vartheta - 3\cos\vartheta\right).
\end{aligned} \tag{4.3}$$

Further: $Y_{l,-m} = (-1)^m\,Y^*_{l,m}$, and thus $q_{l,-m} = (-1)^m\,q^*_{l,m}$.

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## 4.2. Current in cylindrical wire

Consider a straight cylindric wire of radius $R$ oriented along the $z$-axis. The magnitude of the current density inside this wire depends on the distance from the center of the wire as follows:

$$j(\rho) = j_0 e^{-\rho^2/R^2} \theta(R - \rho)\,, \tag{4.4}$$

where $\rho = \sqrt{x^2 + y^2}$ and $\theta(x)$ is the unit step function.

**a)** Find the total current $I$ flowing through the wire. Express $j_0$ through $I$.

**b)** Find the magnetic field inside and outside the wire as a function of the total current. Sketch the field lines, paying attention to the direction. Let the current flow into the positive $z$-direction.

## 4.3. Magnetic field of a finite coil

Consider a wire coiled up cylindrically along the $z$-axis. Let $R$ be the radius of this cylindric coil and $L$ its length (it starts at $z = -L/2$ and ends at $z = +L/2$). Let $n = N/L$ be the winding number per unit length and $I$ the (constant) current flowing through the wire. You may neglect boundary effects.

Calculate the $z$-component of the magnetic flux density $B$ for points on the symmetry axis. Determine the magnetic field for $L \to \infty$ at constant $n$.

## 4.4. Rotation gymnastics

The rotation group (in $N$ dimensions) is defined starting from the set of linear mappings of a vector space which leave the canonical scalar product invariant.

**a)** Prove that a linear transformation which leaves the norm of all vectors invariant also conserves the scalar product between two arbitrary vectors. Then show that any matrix preserving the norm of all vectors is orthogonal.

**b)** Show that the determinant of any orthogonal matrix is either $+1$ or $-1$.

Orthogonal matrices with negative determinant represent transformations that involve reflection. Since we are interested in rotations, let us restrict ourselves to the group of matrices with positive determinant, i.e. the special orthogonal group SO($N$).

**c)** Write down the matrices that represent rotations of an infinitesimal angle $d\varphi$ around the $i$-th axis in three dimensions and subtract the identity from each of them. Find a simple expression of the resulting matrices in terms of the totally antisymmetric tensor $\varepsilon_{ijk}$.

**d)** Show that infinitesimal rotations commute with each other up to higher order terms, whereas macroscopic rotations do not commute in general.

**e)** Write down the infinitesimal rotation of angle $d\varphi$ around a generic unit vector $\vec{n}$ (use the fact that $\vec{n}$ is left unchanged). By performing a large number of such rotations, extend the result to macroscopic angles $\varphi$ around $\vec{n}$. Show that for every rotation with $\varphi \in (0, 2\pi)$ and arbitrary $\vec{n}$, there exists another representation with different $\varphi'$, $\vec{n}'$.

## 5.1. Magnetic moment of a rotating spherical shell

A spherical shell of radius $R$ and charge $Q$ (homogeneously distributed on the surface) is rotating around its $z$-axis with angular velocity $\vec{\omega} = \omega\vec{e}_z$.

**a)** Calculate the current density $\vec{j}(\vec{r}) = \vec{v}(\vec{r})\rho(\vec{r})$.

**b)** Calculate the magnetic moment $\vec{m} = \frac{1}{2}\int d^3r(\vec{r} \times \vec{j}(\vec{r}))$ of the spherical shell.

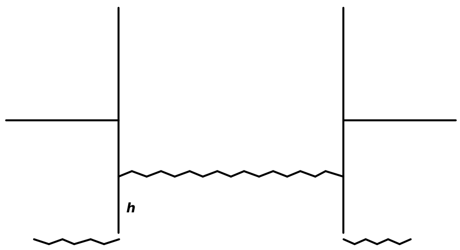
**c)** Show that the leading behaviour of the magnetic field generated by this sphere for $|\vec{r}| \gg R$ is that of a magnetic dipole, and write down the leading term of $\vec{B}$.

*Hint*: Use Biot–Savart law and keep only the leading non-vanishing terms in $R/|\vec{r}|$.

**d)** Now let $\vec{r}_2$ be a vector such that $\vec{r}_2 \perp \vec{\omega}$ and $|\vec{r}_2| \gg R$. Calculate the lowest order contribution of the force exerted by the magnetic field from the previous part on another identical sphere placed at a point $\vec{r}_2$ and rotating with angular velocity $\vec{\omega}_2$ parallel to $\vec{\omega}$. Due to the large distance between the spheres you can approximate them as two point-like objects carrying some magnetic moment.

## 5.2. Capacitor filled with dielectric

Consider a parallel plate capacitor with quadratic plates of edge length $a$ and distance $d$ between the plates. It is charged to the amount $\pm Q$ and subsequently separated from the voltage source. When the charged capacitor is placed on top of a dielectric fluid (with density $\rho_{\text{fl}}$, permittivity $\varepsilon_r$), the fluid rises between the plates up to a maximal height $h_0$.




**a)** Find the electrostatic energy $W_{\text{el}}(h)$ stored in the capacitor as a function of the height of the raised fluid $h$ and the parameters defined above.

**b)** Find the potential energy $W_{\text{pot}}(h)$ of the fluid between the plates as a function of $h$.

**c)** Derive the defining equation for $h_0$ under the assumption that the total energy is minimized. Which amount of charge must be taken to the capacitor, so that the fluid rises up to half of the total height? Assume that $a = 20\,\text{cm}$, $d = 5\,\text{mm}$, $\varepsilon_{\text{r}} = 3$ and $\rho_{\text{fl}} = 0.8\,\text{g/cm}^3$.

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## 5.3. Iron pipe in a magnetic field

An infinitely long hollow cylinder (inner radius $b$, outer radius $a$) is placed with its axis orthogonally to the initially homogeneous magnetic field $\vec{B}_0$. The hollow cylinder is made of iron (permeability $\mu$). The initial field $\vec{B}_0$ can be assumed to be sufficiently small not to saturate the iron, and the permeability $\mu$ is constant in the region of interest.

**a)** Derive the expression for $\vec{B}$ in the cavity ($r < b$).

*Hint*: Use the absence of free currents to describe the magnetic field $\vec{H}$ by means of a scalar potential $\Phi$ via $\vec{H} = -\vec{\nabla}\Phi$. The Laplace equation holds in all three relevant regions of space. In cylindrical coordinates $(r, \varphi)$ the Laplacian reads

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} \,. \tag{5.1}$$

The boundary conditions of $\vec{H}$ and $\vec{B}$ at the surfaces as well as the behaviour for $|\vec{r}| \to 0$ and $|\vec{r}| \to \infty$ fix the constants in the solution of the differential equation.

**b)** Sketch the magnetic field lines in the full region of space, before and after the cylinder has been placed in the field. Consider also the cases of a paramagnetic ($\mu_\mathrm{r} > 1$), a diamagnetic ($\mu_\mathrm{r} < 1$), and a superconducting ($\mu_\mathrm{r} = 0$) cylinder.

## 6.1. Green's functions in electrostatics

In this problem we analyse the electrostatic Green's function in more detail. We consider Green's functions on a volume $V$ with Dirichlet and Neumann boundary conditions on the surface $\partial V$.

**a)** Apply Green's second identity,

$$\int_V d^3x \big(\phi \vec{\nabla}^2 \psi - \psi \vec{\nabla}^2 \phi\big) = \oint_{\partial V} d^2x\, \vec{n}\cdot\big(\phi \vec{\nabla}\psi - \psi \vec{\nabla}\phi\big) \tag{6.1}$$

with $\phi = G(\vec{y}, \vec{x})$ and $\psi = G(\vec{z}, \vec{x})$. Use that $\vec{\nabla}^2_x G(\vec{y}, \vec{x}) = -\delta^{(3)}(\vec{x} - \vec{y})$. Express the difference $G(\vec{y}, \vec{z}) - G(\vec{z}, \vec{y})$ in terms of an integral over the surface $\partial V$.

**b)** Show that a Green's function $G_\mathrm{D}(\vec{x}, \vec{y})$ with Dirichlet boundary conditions $G_\mathrm{D}(\vec{x}, \vec{y}) = 0$ for $\vec{y} \in \partial V$ must be symmetric in $\vec{x}$ and $\vec{y}$.

**c)** Argue that $\vec{n}_y\cdot\vec{\nabla}_y G_\mathrm{D}(\vec{x}, \vec{y}) \to -\delta^2(\vec{x} - \vec{y})$ for $\vec{x} \to \partial V$ and $\vec{y} \in \partial V$. For the case $\vec{x} \not\to \vec{y}$ you can use the Dirichlet boundary condition for $G_\mathrm{D}(\vec{x}, \vec{y})$. To understand the special case $\vec{x} \to \vec{y}$, integrate the above expression over all $\vec{y} \in \partial V$ before performing the limit.

**d)** Consider the Neumann boundary condition $\vec{n}_y\cdot\vec{\nabla}_y G_\mathrm{N}(\vec{x}, \vec{y}) = -F(\vec{y})$ for $\vec{y}$ in $\partial V$ with $\oint_{\partial V} d^2x\, F(\vec{x}) = 1$. Show that $G_\mathrm{N}(\vec{x}, \vec{y})$ is not symmetric in $\vec{x}$ and $\vec{y}$ in general. Construct a Green's function $G'_\mathrm{N}(\vec{x}, \vec{y}) = G_\mathrm{N}(\vec{x}, \vec{y}) + H(\vec{y}) + K(\vec{x})$ that is symmetric in $\vec{x}$ and $\vec{y}$. What properties must $H$ and $K$ have?

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## 6.2. Green's function in two dimensions

**a)** In this problem we consider Green's functions in two dimensions. Consider a Green's function inside the square $0 \le x \le 1$, $0 \le y \le 1$ with Dirichlet boundary conditions on the edges of the square. Show that the Green's function which satisfies the boundary conditions has the expansion

$$G(x, y, x', y') = 2 \sum_{n=1}^{\infty} g_n(y, y') \sin(n\pi x) \sin(n\pi x') \ . \tag{6.2}$$

The coefficients $g_n(y, y')$ satisfy the conditions

$$\left( \frac{\partial^2}{\partial y^2} - n^2\pi^2 \right) g_n(y, y') = -\delta(y' - y) \ , \tag{6.3}$$

$$g_n(y, 0) = g_n(y, 1) = 0 \ . \tag{6.4}$$

Use the identity for $0 \le x, x' \le 1$

$$\sum_{n=1}^{\infty} \sin(n\pi x) \sin(n\pi x') = \tfrac{1}{2}\delta(x - x') \, . \tag{6.5}$$

**b)** The homogeneous solution of the differential equation for $g_n$ can be expressed in terms of a linear combination of $\sinh(n\pi y')$ and $\cosh(n\pi y')$. Make an ansatz in the two regions $y < y'$ und $y > y'$ that satisfies the boundary conditions and takes into account the discontinuity of the first derivative at $y = y'$ induced by the source delta function. Show that the explicit form of $g_n$ is

$$g_n(x, y, x', y') = \frac{\sinh(n\pi y_<) \sinh(n\pi(1 - y_>))}{\pi n \sinh(n\pi)} \ , \tag{6.6}$$

where $y_<$ ($y_>$) is the smaller (larger) value of $y$ and $y'$.

**c)** Let the square now have a uniform charge density $\rho$. Furthermore assume that the edges bounding the square are grounded. Use the Green's function determined above to show that the potential is

$$\begin{aligned} \Phi(x, y) = {} & \frac{4\rho}{\pi^3 \varepsilon_0} \sum_{m=0}^{\infty} \frac{\sin((2m+1)\pi x)}{(2m+1)^3} \\ & \cdot \left( 1 - \frac{\sinh((2m+1)\pi(1-y)) + \sinh((2m+1)\pi y)}{\sinh((2m+1)\pi)} \right) \ . \end{aligned} \tag{6.7}$$

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## 6.3. Cartesian multipole expansion

In this exercise we review the concept of cartesian multipole expansion, which is used to decompose the integral expression of the potentials $\phi(\vec{x})$ and $\vec{A}(\vec{x})$, as well as the fields $\vec{E}(\vec{x})$ and $\vec{B}(\vec{x})$ created by localised static charge and current densities $\rho(\vec{y})$ and $\vec{j}(\vec{y})$ into several components. These components have important physical meanings and are of different relevance in the regime of $\vec{x}$ being far away from the source.

We assume that the sources of the electric/magnetic field are contained in an area $A$ of extension $a$ around the origin. The corresponding scalar potential $\Phi$ generated by a given charge density $\rho(\vec{y})$ can be written as

$$\Phi(\vec{x}) = \int d^3y\, \rho(\vec{y})\, \frac{1}{4\pi\varepsilon_0|\vec{x}-\vec{y}|}\ . \tag{6.8}$$

One expects that in the region $|\vec{x}| \gg a$ the scalar potential essentially looks like the potential created by a point charge placed somewhere in $A$ carrying the charge $q = \int d^3y\rho(\vec{y})$. We will reproduce this behaviour by expanding the term $1/|\vec{x}-\vec{y}|$ for fixed $\vec{x}$ in a Taylor series of $\vec{x}$ around the point $\vec{y} = 0$. In the limit $|\vec{x}| \to \infty$ this choice of $\vec{y}$ will drop out, however for finite $\vec{x}$ this choice matters.

**a)** Calculate the Taylor expansion of $1/|\vec{x}-\vec{y}|$ up to $\mathcal{O}(1/|\vec{x}|^4)$, and use the result to expand the electric potential of a charge distribution as

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0|\vec{x}|}\left[Q + \frac{x_i}{|\vec{x}|^2}\,Q_i + \frac{x_ix_j}{2|\vec{x}|^4}\,Q_{ij} + \frac{x_ix_jx_k}{6|\vec{x}|^6}\,Q_{ijk}\right] + \mathcal{O}\big(1/|\vec{x}|^5\big). \tag{6.9}$$

Determine the total charge $Q$, the dipole moment $Q_i$, the quadrupole moment $Q_{ij}$ and the octupole moment $Q_{ijk}$.

**b)** Now we consider a concrete charge distribution $\rho$ that is generated by a finite dipole consisting of a charge $+\lambda q$ at $(0,0,1/(2\lambda))$ and a charge $-\lambda q$ at $(0,0,-1/(2\lambda))$. Calculate $Q$, $Q_i$, $Q_{ij}$ and $Q_{ijk}$. Show that only in the limit $\lambda \to \infty$, we obtain a perfect dipole (i.e. that all other moments of the multipole expansion vanish).

**c)** The cartesian components $A_i$ of the vector potential can be calculated in the same way as the scalar potential, i.e.

$$A_i(\vec{x}) = \mu_0 \int d^3y\, j_i(\vec{y})\, \frac{1}{4\pi|\vec{x}-\vec{y}|}\ . \tag{6.10}$$

Show that the first term in the multipole expansion vanishes in case of a time independent current density, and calculate the vector potential up to the first order.

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## 6.4. Multipole moments in cartesian and spherical coordinates

In this problem we will analyse the relation between cartesian and spherical quadrupole moments. Consider the cartesian quadrupole moment

$$Q_{ij} = Q_{ji} = \int d^3x\, \rho(\vec{x})\, (3x_i x_j - \delta_{ij}\vec{x}^2)\,, \tag{6.11}$$

and the spherical quadrupole moments

$$q_{lm} = \int dr\, d\vartheta\, d\varphi\, \sin\vartheta\, \rho(\vec{x})\, r^{2+l}\, Y^*_{l,m}(\vartheta, \varphi)\,. \tag{6.12}$$

Express the cartesian multipole moments $Q_{ij}$ through the spherical ones for $l = 2$ and use the following identities,

$$\begin{aligned}
Y_{2,\pm 2}(\vartheta, \varphi) &= \sqrt{\frac{15}{8}}\, \sin^2\vartheta\, e^{\pm 2i\varphi}\,,\\
Y_{2,\pm 1}(\vartheta, \varphi) &= \mp\sqrt{\frac{15}{2}}\, \cos\vartheta\, \sin\vartheta\, e^{\pm i\varphi}\,,\\
Y_{2,0}(\vartheta, \varphi) &= \sqrt{\frac{5}{4}}\left(3\cos^2\vartheta - 1\right)\,.
\end{aligned} \tag{6.13}$$

## 6.5. Magnetic field in a non-coaxial cylinder cavity

Consider a conducting cylinder of radius $a$, with a cylindrical cavity of radius $b$ whose axis is parallel to the axis of the conducting cylinder. The distance between the axes is $d$, and $d + b < a$. The current density $\vec{j}$ is uniform through the conductiong part of the cylinder. Calculate the magnetic field $\vec{B}$ inside the cavity.

*Hint*: Use the superposition principle and Ampère's law,

$$\oint_{\partial S} \vec{B}\cdot d\vec{r} = \mu_0 \int_S d^2x\, \vec{n}\cdot\vec{j}\,. \tag{6.14}$$

## 6.6. Magnetic field of a circular loop

Consider a conducting wire forming a circle of radius $R$ in the centre of the $x$-$y$-plane. A constant current $I$ flows counterclockwise through this loop.

**a)** Calculate the magnetic field $\vec{B}$ at some point on the $z$-axis.

**b)** Now calculate the magnetic field $\vec{B}$ at an arbitrary point in the $x$-$y$-plane.

## 6.7. Magnetic field of a kinked wire

An infinite wire carrying a constant current $I$ runs along the positive $y$-axis, kinks at the origin, and then runs along the positive $x$-axis. Show that the magnetic field $\vec{B}$ in the $x$-$y$-plane for $x, y > 0$ is given by

$$\vec{B} = \frac{\mu_0 I}{4\pi}\left(\frac{1}{x} + \frac{1}{y} + \frac{x}{y\sqrt{x^2+y^2}} + \frac{y}{x\sqrt{x^2+y^2}}\right)\vec{e}_z\,. \tag{6.15}$$

## 7.1. Charged particle in an electromagnetic field

Consider a point particle carrying charge $q$ in an electromagnetic field, described by a vector potential $\vec{A}$ and a scalar potential $\Phi$. The Lagrangian of the particle is given by

$$L(\vec{x}, \dot{\vec{x}}, t) = \tfrac{1}{2} m \dot{\vec{x}}^2 + q \dot{\vec{x}} \cdot \vec{A}(\vec{x}, t) - q\Phi(\vec{x}, t)\,, \tag{7.1}$$

where $\vec{x}$ is the position of the particle and $m$ is its mass.

**a)** Determine the canonical momentum $\vec{p}$,

$$\vec{p} = \frac{\partial L(\vec{x}, \dot{\vec{x}}, t)}{\partial \dot{\vec{x}}}\,. \tag{7.2}$$

What is the relation between the canonical momentum $\vec{p}$ and the kinetic momentum $m\dot{\vec{x}}$? Perform a Legendre transformation to determine the Hamiltonian,

$$H(\vec{x}, \vec{p}, t) = \vec{p} \cdot \dot{\vec{x}} - L(\vec{x}, \dot{\vec{x}}, t)\,. \tag{7.3}$$

**b)** Use $\vec{B} = \vec{\nabla} \times \vec{A}$ to explicitly verify

$$\sum_{j=1}^{3} \left( \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) \dot{x}_j = \left( \dot{\vec{x}} \times \vec{B} \right)_i . \tag{7.4}$$

**c)** Start from the Hamiltonian equations

$$\dot{p}_i = -\frac{\partial H}{\partial x_i}\,, \qquad \dot{x}_i = \frac{\partial H}{\partial p_i} \tag{7.5}$$

and derive the equation of motion for the charged particle in an electromagnetic field

$$m\ddot{\vec{x}} = q\left( \vec{E} + \dot{\vec{x}} \times \vec{B} \right)\,. \tag{7.6}$$

## 7.2. Induction in a magnetic field

A homogeneous magnetic field $\vec{B}$ is aligned along the $z$-axis. Within this magnetic field, a conducting wire forming a circle of radius $R$ rotates with circular velocity $\vec{\omega}$. Its rotational axis lies in the plane of the conductor and passes through its centre. Let $\varphi$ be the angle between the rotational axis and the field direction. Find the induced voltage in the conductor as a function of time.

## 7.3. Self-induction of a coaxial cable

A coaxial cable is represented by two coaxial conducting cylindrical shells with radii $R_1$ and $R_2$ with $R_1 < R_2$. A current $I$ is flowing through each of the cylindrical shells along their axes in opposite directions. Calculate the self-induction per unit length of this coaxial cable.

*Hint:* First calculate the magnetic field of the cable. Then determine its self-induction from the magnetic energy of the cable via

$$W_\mathrm{m} = \tfrac{1}{2} L I^2\,. \tag{7.7}$$

## 8.1. Fourier transform

The Fourier transformation and its inverse are given by

$$f(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, \tilde{f}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}\,, \qquad \tilde{f}(\vec{k}) = \int d^3x\, f(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}\,. \tag{8.1}$$

Calculate the Fourier transform of the following functions/equations. The functions $f$ and $g$ have Fourier transforms $\tilde{f}$ and $\tilde{g}$.

**a)** $af(\vec{x}) + bg(\vec{x}) \quad (a, b \in \mathbb{C})$,

**b)** $\vec{\nabla} f(\vec{x})$,

**c)** $f(\vec{x})\, g(\vec{x})$,

**d)** $f(\vec{x}) = f^*(\vec{x})$,

**e)** $\delta^3(\vec{x})$,

**f)** $\vec{\nabla}\delta^3(\vec{x})$.

## 8.2. Elliptically polarized waves

A wave $\vec{E}(\vec{x}, t)$ with the wave vector $\vec{k} = k\vec{e}_z$ is given by

$$\begin{aligned} E_x(\vec{x}, t) &= A\cos(kz - \omega t)\,, \\ E_y(\vec{x}, t) &= B\cos(kz - \omega t + \varphi)\,. \end{aligned} \tag{8.2}$$

**a)** The path of the vector $\vec{E}(0, t)$ describes the polarization of the wave. Show that it is an ellipse. For which values of $A$, $B$ and $\varphi$ is it a circle?
*Hint*: The equation of an ellipse is given by

$$aE_x^2 + 2bE_xE_y + cE_y^2 + f = 0\,, \tag{8.3}$$

where $b^2 - ac > 0$ and $f < 0$ .

**b)** Show that for general $A$ and $B$ the wave could be written as a superposition of two opposite circularly polarized waves,

$$\vec{E}(\vec{x}, t) = \vec{E}_+(z, t) + \vec{E}_-(z, t)\,, \tag{8.4}$$

where $E_\pm(z, t) = \mathrm{Re}(A_\pm \vec{e}_\pm e^{i(kz-\omega t)})$. $A_\pm$ is a constant and $\vec{e}_\pm = \frac{1}{\sqrt{2}}(\vec{e}_x + i\vec{e}_y)$. Determine $A_\pm$ as a function of $A$, $B$, and $\varphi$.
*Hint:* Write the wave as the real part of a complex vector and express $\vec{e}_x$ and $\vec{e}_y$ as functions of $\vec{e}_\pm$.

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## 8.3. Group velocity

A Gaussian wave packet $\phi(x,t)$ is moving in a dispersive medium (i.e. $\omega(k)$ depends non-linearly on $k$). At time $t = 0$ it is given by

$$\phi(x, t = 0) = \exp\left(-\frac{x^2}{2(\Delta x)^2}\right), \tag{8.5}$$

where we consider $\Delta x$ as a measure for the spatial extent of the wave packet. The time dependency is given by

$$\phi(x,t) = \mathrm{Re} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{\phi}(k)\, e^{ikx - i\omega(k)t}, \tag{8.6}$$

where $\tilde{\phi}(k)$ is the Fourier transform of $\phi(x, t = 0)$ .

**a)** Show by completing the squares that the Fourier transform wave packet at $t = 0$ has a Gaussian profile. What is the relation between $\Delta x$ and the analogous $\Delta k$? What does this mean?

*Hint*:

$$\int_{-\infty}^{\infty} dx\, \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = \sqrt{2\pi}\, |\sigma|\,. \tag{8.7}$$

**b)** Show that the maximum of the wave packet covers a distance of $v_g t$ in the time interval $t$, where the *group velocity* $v_g$ is given by

$$v_g = \left.\frac{d\omega}{dk}\right|_{k_0}. \tag{8.8}$$

Here, $k_0$ denotes the wave number at the maximum of $\tilde{\phi}(k)$.

*Hint:* Expand $\omega(k)$ to the first order in $k$ around $k_0$, and evaluate the change in the maximum of the wave packet using (8.6).

**c)** What is the speed of the individual phase? Under which circumstances do phase velocity and group velocity of the wave coincide?

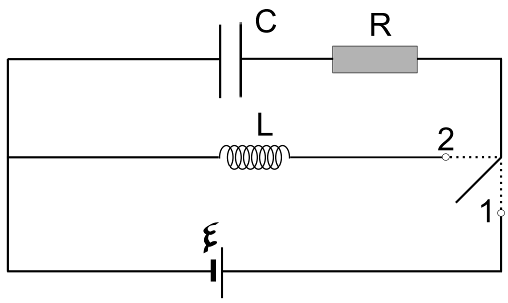
**d)** Estimate how fast the wave packet is widening by finding an expression for the variation of the group velocity inside the pulse. Use the relation between $\Delta k$ and $\Delta x$ from part a), and interpret the result accordingly.

*Hint:* Estimate the variation as the difference between the group velocities at $k_0$ and $k_0 + \Delta k$ (analoguously to (8.8)), and determine $\Delta v_g$ from an expansion of $\omega(k)$ around $k_0$ up to the first contributing order.

## 9.1. Dynamics of an electric circuit

Consider the circuit shown in (9.1). The circuit consists of a voltage source $\xi$, a resistor of resistance $R$, a capacitor of capacitance $C$, and a solenoid of inductance $L$.

Note: The laws for the three devices are: $U = RI$, $Q = CU$, $U = L\dot{I}$, respectively.




(9.1)

**a)** When the switch is in position 1, the voltage source, resistor and capacitor form a circuit. Assume that capacitor is initially uncharged. For the switch in position 1, write down a differential equation for the charge on the capacitor, solve it, and calculate the time it takes the capacitor to charge to $(1 - 1/e)$ of its maximal capacity.

When the switch is in position 2, the solenoid, resistor and capacitor form a circuit. Assume that the capacitor is fully charged when the switch is set to position 2 (time $t = 0$). Correspondingly, there are no currents flowing in the circuit at time $t = 0$.

**b)** For the switch in position 2, write down a differential equation for the charge on the capacitor in the case $R = 0$, solve it, and find the natural frequency of the the circuit.

**c)** Now set $R > 0$. Write down the corresponding differential equation, solve it, and sketch the charge on the capacitor as a function of time. Discuss the three different cases that can arise.

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## 9.2. The Poynting vector

Maxwell's equations in vacuum are given by

$$\vec{\nabla} \times \vec{E} = -\dot{\vec{B}}\,, \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \varepsilon_0 \mu_0 \dot{\vec{E}}\,, \tag{9.2}$$

and the speed of light is $c = 1/\sqrt{\varepsilon_0 \mu_0}$.

**a)** Proove the following identity,

$$\frac{1}{2}\frac{\partial}{\partial t}\big(c^2 \vec{B}^2 + \vec{E}^2\big) = -c^2 \vec{\nabla}\cdot\big(\vec{E} \times \vec{B}\big) - \frac{1}{\varepsilon_0}\,\vec{E}\cdot\vec{j}\,. \tag{9.3}$$

**b)** Consider a particle with charge $q$ moving in the electromagnetic field with velocity $\vec{v}$. Show that the derivative of its kinetic energy is given by

$$\dot{W}_{\text{kin}} = q\vec{v}\cdot\vec{E}\,. \tag{9.4}$$

What is the equivalent for a continuous charge distribution?

The Poynting vector is defined as

$$\vec{S} = \varepsilon_0 c^2 \vec{E} \times \vec{B}\,. \tag{9.5}$$

**c)** Prove Poynting's theorem,

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\varepsilon_0 \int_V d^3x\big(c^2\vec{B}^2 + \vec{E}^2\big) + W_{\text{kin}}\right) = -\int_{\partial V} d^2x\,\vec{n}\cdot\vec{S}\,, \tag{9.6}$$

where $V$ is some volume and $\partial V$ its surface. For $\dot{W}_{\text{kin}}$, insert your result from b). Interpret the physical meaning of each of the terms.

**d)** Show that the time-averaged Poynting vector for a plane wave in a non-conducting medium can be written as

$$\langle \vec{S} \rangle = \tfrac{1}{2}\varepsilon_0 c^2\, \text{Re}\big(\vec{E}_0 \times \vec{B}_0^*\big)\,, \tag{9.7}$$

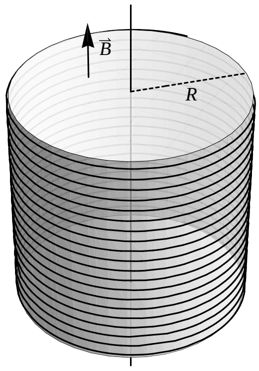
where $\vec{E}_0$ and $\vec{B}_0$ are the (complex) amplitudes of the electric and magnetic fields with the time dependency $e^{i\omega t}$,

$$\vec{E}(t) = \text{Re}(\vec{E}_0 e^{i\omega t})\,, \qquad \vec{B}(t) = \text{Re}(\vec{B}_0 e^{i\omega t})\,. \tag{9.8}$$

*Hint:* Compute the average value over a period $T$,

$$\langle \vec{S} \rangle = \frac{1}{T}\int_0^T \vec{S}(t)\,dt\,. \tag{9.9}$$

## 10.1. Feynman's paradox




(10.1)

Consider a long straight wire along the $z$-axis carrying a uniform linear charge density $\lambda$, surrounded by an equally long cylindrical shell of nonconducting material which bears a uniform surface charge density $\sigma$. Let the radius of the shell be $R$, and let the values of $\sigma$ and $\lambda$ be chosen in such a way that the net electric charge per unit length of the system vanishes. Moreover, a conducting wire (covered with an insulator) is coiled around the cylindrical shell with number of turns per length $n$. The shell is free to rotate without friction around the symmetry axis and has a mass per unit length equal to $\rho$, including the wire.

The system is initially at rest, with a current $i_0$ flowing through the solenoid. The current inside the coil is then decreased until it vanishes, e.g. because it is powered by a battery at the end of its lifetime.

**a)** Compute the torque $d\vec{M}$ exerted on a slice of the cylindrical shell of height $dz$ by the non-conservative electric field due to the changing current.

**b)** Compute the magnetic field generated by a cylindrical shell with surface charge density $\sigma$ that rotates with angular speed $\omega$ around its symmetry axis.

**c)** Compute the final angular frequency $\omega_{\mathrm{f}}$ of the system by integrating the torque over time. Account for the final flux of magnetic field due to the motion of charges.

The considered situation seems to lead to a paradox: there are no external torques acting on the cylindrical shell, yet the angular momentum appears to change from being zero to some finite value during the experiment. The second part of the exercise is meant to resolve this apparent contradiction.

**d)** Find the Poynting vector $\vec{S}$ everywhere for a generic static, constant magnetic field $\vec{B}$ along the $z$-axis inside the cylinder; then use it to compute the angular momentum associated with the electromagnetic field configuration.

**e)** Write down the appropriate angular momentum conservation law and use it to cross-check your result for $\omega_{\mathrm{f}}$. Explain why the argument that led to inconsistency does not hold.

$\longrightarrow$

## 10.2. Plane waves in a conductive medium

**a)** We consider plane waves in free space. Derive the wave equations for the electromagnetic field from the Maxwell equations. Show that the plane wave,

$$\begin{Bmatrix} \vec{E}(\vec{x},t) \\ \vec{B}(\vec{x},t) \end{Bmatrix} = \mathrm{Re}\left[\begin{Bmatrix} \vec{E}_0 \\ \vec{B}_0 \end{Bmatrix} \exp\bigl(i\vec{k}\cdot\vec{x} - i\omega t\bigr)\right] , \tag{10.2}$$

represents a solution of these equations and derive the dispersion relation. What is the wave velocity?

**b)** Now consider a medium with finite conductivity $\sigma > 0$. Here the current density and the electric field are connected by $\vec{j} = \sigma\vec{E}$. Find the wave equation for the electromagnetic field in this case and show that the amplitude of a plane wave decays with the penetration depth into the medium.

**c)** For a low-frequency wave, calculate the penetration depth $\delta$ characterising the depth to which a plane wave can enter a conductive medium. $\delta$ is conventionally defined as the distance at which the amplitude of the plane wave decays by a factor $e$.

## 11.1. Rectangular waveguide

Consider an waveguide extended infinitely along the $z$-axis with a rectangular basis $0 < x < d_x$ and $0 < y < d_y$. Its surfaces are ideal conductors. Due to the geometry of the problem, you can make the following ansatz for propagating electromagnetic waves,

$$\begin{aligned}\vec{E}(x,y,z,t) &= \operatorname{Re}\big(\vec{E}_0(x,y)\,e^{i(kz-\omega t)}\big)\,,\\ \vec{B}(x,y,z,t) &= \operatorname{Re}\big(\vec{B}_0(x,y)\,e^{i(kz-\omega t)}\big)\,.\end{aligned} \tag{11.1}$$

**a)** The 3D-vectors split into 2D-vectors (here: $x$- and $y$-components) and scalars ($z$-component). From the Maxwell equations, derive equations for the $x$- and $y$-components of $\vec{E}_0$ and $\vec{B}_0$ in terms of their $z$-components, and show the following equations to hold for the $z$-components,

$$\begin{aligned}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]E_z &= 0\,,\\ \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]B_z &= 0\,.\end{aligned} \tag{11.2}$$

**b)** Express the boundary conditions $E_\parallel = B_\perp = 0$ as conditions for the $z$-components of the fields.

**c)** Determine the solutions for so-called transverse magnetic waves (TM waves), which satisfy $B_z = 0$.

**d)** Show that no transverse electromagnetic (TEM) waves (i.e. waves with $E_z = B_z = 0$) exist in an ideal waveguide.

*Hint:* Use Gauss' theorem and Faraday's law as well as the boundary condition for $E_\parallel$ to show that there are no TEM waves in this waveguide.

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## 11.2. Dipole radiation

A thin, ideal conductor connects two metal balls at positions $z = \pm a$. The charge density is

$$\rho(\vec{x}, t) = Q\,\delta(x)\delta(y)\big[\delta(z-a) - \delta(z+a)\big]\cos(\omega t) \tag{11.3}$$

with $a$, $Q$ and $\omega$ constant. The current between the two metal balls flows along the wire.

**a)** Calculate the time average of the angular distribution of the radiation power $\langle dP/d\Omega\rangle$.
*Hint:* Calculate first the vector potential $\vec{A}(\vec{x}, t)$ from the current density $\vec{j}$ and use simplifying approximations.

**b)** Calculate $\langle dP/d\Omega\rangle$ in dipole approximation using the formula

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle |\ddot{\vec{p}}|^2 \rangle \frac{\sin^2\vartheta}{16\pi^2\varepsilon_0 c^3}\,. \tag{11.4}$$

## 12.1. Optics via the principle of least action

Fermat's principle states that light travelling between two points in space $\vec{x}_1$ and $\vec{x}_2$ should minimise the optical path. The latter is given by

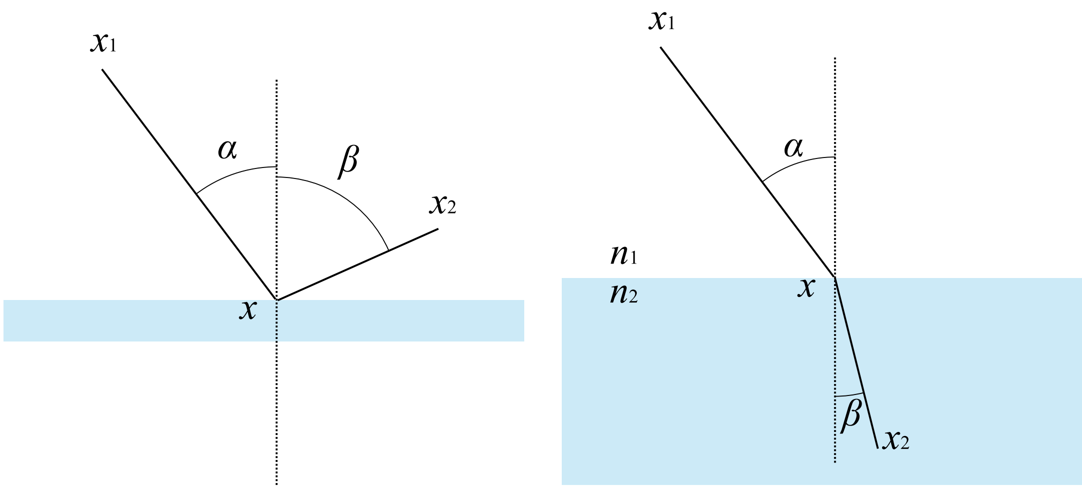
$$S = \int_{\vec{x}_1}^{\vec{x}_2} n(\vec{x})\, dl, \tag{12.1}$$

where $n(\vec{x})$ denotes the refractive index of the matter and $dl = \sqrt{dx^2 + dy^2 + dz^2}$ is the length of the infinitesimal element of the trajectory connecting $\vec{x}_1$ to $\vec{x}_2$. This can be directly interpreted as the principle of least action.

*Hint:* It is convenient to parametrise the trajectory for this integral, namely

$$S = \int_{t_1}^{t_2} n(\vec{x}(t))\, \frac{dl}{dt}\, dt = \int_{t_1}^{t_2} n(\vec{x}(t)) \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2}\, dt\,. \tag{12.2}$$

**a)** Find the trajectory of light between two points in a homogeneous medium.




(12.3)

**b)** Now consider light that is reflected from a plane mirror. The light travels in vacuum from point $\vec{x_1}$ to some point $\vec{x}$ on the surface of the mirror, and then, again in vacuum, to some point $\vec{x_2}$. Minimise the action over all positions of $\vec{x}$ on the mirror, and compare the incident and emergent angles for the chosen value of $\vec{x}$.

**c)** Finally, consider light propagating between two points in space which are located in different media with the refractive indices $n_1$ and $n_2$, respectively. The boundary surface between the two media is a plane. Consider a light path between point $\vec{x}_1$ in the medium with $n_1$ and $\vec{x}_2$ in the medium with $n_2$, which passes through the point $\vec{x}$ on the boundary surface. Choose $\vec{x}$ that minimises the total light path. Find the relationship between incident and emergent angles (Snell's law).

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## 12.2. Refraction of planar waves

A planar wave is incident perpendicularly onto a planar layer between two media. The indices of refraction of the three non-magnetic layers are $n_1$, $n_2$ and $n_3$. The thickness of the central layer $d$, while the other two media each fill half spaces.

**a)** Calculate the reflection and transmission coefficients (i.e. the ratio of the reflected and transmitted wave with the incoming energy flux).

*Hint:* The time-averaged energy-flux density of a complex wave is given by

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0} \mathrm{Re}(\vec{E} \times \vec{B}^*) \,. \tag{12.4}$$

**b)** Let the medium with index $n_1$ be part of an optical system (e.g. a lens), and the medium with index $n_3$ be the air ($n_3 = 1$). The surface of the first medium, should have a layer of the medium with index $n_2$ of such a thickness that for a given frequency $\omega_0$, there is no reflected wave. Determine the thickness $d$ and the index of refraction $n_2$ of this layer.

## 12.3. Scattering of light

Classical light-scattering theory (known as a Rayleigh theory) is used to describe light being scattered off small molecules (with an extension much smaller than the wavelength $\lambda$ of the light). Here we consider electric and magnetic fields and intensity of light scattered off small particles.

**a)** First consider a plane monochromatic light wave propagating in $x$ direction, which is polarised in $z$-direction. This wave is scattered off a small polarisable, but non-magnetic particle at the origin. The incident wave induces a dipole moment to the particle, that is proportional to the local field, $\vec{p}(t) = \alpha \vec{E}(0, t)$, where $\alpha$ is its polarisability. Determine the electric and magnetic fields of the scattered wave at a far-away point $\vec{x}$, depending on the incident field $E_0$, the distance from the origin $r$, and the angle $\vartheta$ between $\vec{x}$ and the $z$-axis.

**b)** Calculate the intensity of this scattered light wave, at a point $\vec{x}$ far away from the scattering particle

*Hint:* Use the Poynting vector.

**c)** Use the wave-length dependence of the intensity of the scattered wave ($\propto 1/\lambda^4$) derived in the previous subproblem, to explain qualitatively the blue colour of the cloudless sky and the red colour of the sunrise and the sunset.

## 13.1. Diffraction through a rectangular slit

A rectangular opening with sides of length $a$ and $b$ ($b \geq a$) with corners at $x = \pm a/2$, $y = \pm b/2$ is located in a flat, perfectly conducting sheet filling the $xy$-plane. A plane wave propagating in $z$-direction with linear polarisation at an angle of $\alpha$ w.r.t. the $y$-axis hits the opening.

**a)** Calculate the diffracted fields and power per unit solid angle with the vectorial Smythe–Kirchhoff relation,

$$\vec{E}(\vec{x}) = \frac{ie^{ikr}}{2\pi r}\,\vec{k} \times \int_A d^2x'\,\vec{n} \times \vec{E}(\vec{x}\,')\,e^{-i\vec{k}\cdot\vec{x}\,'}\,, \tag{13.1}$$

assuming that the tangential electric field in the opening is the unperturbed incident field.

**b)** Calculate the corresponding result with the scalar Kirchhoff approximation.

## 13.2. Invariant distance

Show that the squared distance $s_{12}^2 = s_{12}^{\mu} s_{12,\mu}$ ($s_{12} = x_1 - x_2$) of two spacetime points $x_1$ and $x_2$, is a Lorentz scalar, i.e. $s_{12} = s'_{12}$. To do so, use a Lorentz boost with arbitrary direction and velocity,

$$t' = \gamma t - \frac{\gamma}{c^2}\,\vec{x}\cdot\vec{v}, \qquad \vec{x}\,' = \vec{x} - \gamma\vec{v}t + (\gamma - 1)\,\frac{\vec{x}\cdot\vec{v}}{\vec{v}^2}\,\vec{v}\,, \tag{13.2}$$

with $\gamma = 1/\sqrt{1 - \vec{v}^2/c^2}$.

**a)** First, bring the above transformation into a matrix form,

$$x'^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu} x^{\nu}\,. \tag{13.3}$$

**b)** Now choose $\vec{v} = (1, 0, 0)v$ and show that $\Lambda$ defines a Lorentz transformation, i.e.

$$\Lambda^{\lambda}{}_{\mu}\eta_{\lambda\sigma}\Lambda^{\sigma}{}_{\nu} = \eta_{\mu\nu}\,. \tag{13.4}$$

**c)** Finally, show that the squared distance is Lorentz invariant.

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## 13.3. Electromagnetic field tensor

The electromagnetic field tensor is defined by

$$F_{\mu\nu} := -\partial_\mu A_\nu + \partial_\nu A_\mu = \begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\ -\frac{1}{c}E_x & 0 & -B_z & +B_y \\ -\frac{1}{c}E_y & +B_z & 0 & -B_x \\ -\frac{1}{c}E_z & -B_y & +B_x & 0 \end{pmatrix} . \tag{13.5}$$

**a)** Show that the electromagnetic field tensor is invariant under the gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi \tag{13.6}$$

for any scalar field $\chi$.

**b)** The dual electromagnetic field tensor is defined by

$$\tilde{F}_{\mu\nu} := \tfrac{1}{2}\, \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} . \tag{13.7}$$

Determine the matrix $\tilde{F}_{\mu\nu}$.

**c)** Calculate the contractions $F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu}\tilde{F}^{\mu\nu}$ and $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$.

## 13.4. Lienard–Wiechert potential

Consider a charge moving straight along the positive $z$-axis with a uniform velocity $v$ starting at $z = 0$ at $t = 0$. Show that its potential is given by

$$\Phi(\vec{x}, t) = \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{(z - vt)^2 + (1 - v^2/c^2)(x^2 + y^2)}} . \tag{13.8}$$