Second Exercise Sheet due to 6. March

Exercise 1 (Operations on the Bloch Sphere) *Bloch sphere is a representation of states of a two level system. A state ρ corresponds to a vector $\vec{n}$ inside a ball via*

$$\rho = \frac{1 + \vec{n} \cdot \vec{\sigma}}{2}.$$

In this exercise we will see examples of operations on the Bloch sphere. In examples below trace out the photons (when relevant) and compute the associated Kraus map of the Bloch sphere (sketch it also graphically!).

i) Hamiltonian Evolution: *A Hamiltonian $H = \vec{B} \cdot \sigma$ generates an evolution*

$$\rho_t = \exp(-iHt)\rho_0 \exp(iHt).$$

Show that this is a rotation of the Bloch sphere around an axis $\vec{B}$ with an angular velocity $2|\vec{B}|$. Is this rotation clockwise or counterclockwise?

In particular

$$\exp(i\frac{\varphi}{2}\hat{n} \cdot \vec{\sigma}) = \cos(\frac{\varphi}{2}) + i\sin(\frac{\varphi}{2})\hat{n} \cdot \vec{\sigma}$$

is a rotation of the Bloch sphere by an angle φ around the axis $\hat{n}$.

ii) Optical Pumping: *Optical pumping is a process in which two levels $|0\rangle$, $|1\rangle$ are coupled to an excited state $|2\rangle$ by a laser tuned to the transition $1 \to 2$. Excited state $|2\rangle$ then spontaneously decay into the $0, 1$ manifold:*

$$\alpha|0\rangle + \beta|1\rangle \xrightarrow{laser} \alpha|0\rangle + \beta|2\rangle \xrightarrow{sp.decay} \alpha|0\rangle|vaccum\rangle + \beta A_0|0\rangle|\phi_0\rangle + \beta A_1|1\rangle|\phi_1\rangle,$$

where A_0, A_1 are probability amplitudes of spontaneous decay to the state $|0\rangle$ and $|1\rangle$ respectively and $|\phi_0\rangle$, $|\phi_1\rangle$ are wave functions of photons emitted upon the respective transition.

Compute the Kraus map of the process, and in particular find out what happens when the process is repeated many times.

iii) Photon Scattering: *We consider one photon scattering between $S^{1/2}$ and $P^{1/2}$ spin manifolds*[1]*, i.e. both $S^{1/2}$ and $P^{1/2}$ are two level systems. First an absorption of a photon induce a $S^{1/2} \to P^{1/2}$ transition and then the spontaneous emission of a photon returns the state to $S^{1/2}$. Each absorption (or emission) of a photon is accompanied by π rotation of the Bloch*

[1] This is relevant for example for $^{88}Sr^+$ see N. Akerman et. al. PRL 109 (2012) .

sphere around the photon polarization direction. In particular, absorption of $\hat{z}$-polarized photon and consequent emission of a photon linearly polarized in $\hat{\lambda}$ direction is a process (see i))

$$\alpha|0\rangle + \beta|1\rangle \rightarrow -\alpha(\hat{\lambda}\cdot\vec{\sigma})|0\rangle|\hat{\lambda}\rangle + \beta(\hat{\lambda}\cdot\vec{\sigma})|1\rangle|\hat{\lambda}\rangle,$$

where $|\hat{\lambda}\rangle$ is a wave function of the emitted photon. Compute the associated Kraus map if the photon is emitted in a superposition of $\hat{y}$ and $\hat{z}$ polarizations, i.e. having the wave function $\frac{1}{\sqrt{2}}(|\hat{y}\rangle + |\hat{z}\rangle)$.