Exercise 1. *BRST transformation*

The path integral for a non-abelian gauge theory is quadratic in the gauge-fixing term:

$$Z \sim \int \mathcal{D}A^a_\mu \, \mathcal{D}\bar{\psi}^i \, \mathcal{D}\psi^i \, \mathcal{D}\bar{\eta}^a \, \mathcal{D}\eta^a exp \left\{ i \int d^4x \left[\mathcal{L}_{YM} + \mathcal{L}_{fermions} + \mathcal{L}_{ghosts} - \frac{1}{2\xi} \left(\mathcal{G}^a[A] \right)^2 \right] \right\} \quad (1)$$

It can be made linear in the gauge-fixing term by introducing a new bosonic field w^a, and one recovers the previous path integral by integrating out the field w^a. In this formalism, the exponent in the path integral is not gauge-invariant, but is invariant under the so-called BRST symmetry. Under this symmetry, the fields transform as

$$\delta_\theta A^a_\mu = -\frac{\theta}{g} D^{ab}_\mu \eta^b \quad (2)$$

$$\delta_\theta \eta^a = \frac{\theta}{2} f^{abc} \eta^b \eta^c \quad (3)$$

Prove that two successive BRST transformations leave the gauge field invariant:

$$\delta_\theta \delta_\theta A^a_\mu = 0 \quad (4)$$

Exercise 2. *BRST Jacobian*

Show that the path integral is invariant under BRST transformations, i.e. show that the Jacobian of the transformation is unity.

(a) Write down the transformation matrix for a BRST transformation

(b) Writing the Jacobian as

$$J = \begin{pmatrix} A & D \\ C & B \end{pmatrix} \quad (5)$$

The determinant of J can be written as:

$$\det J = \frac{\det A}{\det(B - CA^{-1}D)} \quad (6)$$

where A and B are commuting matrices of the form $\mathbb{1} + \theta M$, where θ is a Grassman variable, and C and D are anticommuting matrices.

Hint. Taylor expand the determinants in powers of θ, remembering that $\theta^2 = 0$ and hence also higher powers vanish. Finally use $det(M) = exp[Tr(ln(M))]$.

Exercise 3. *The BRST charge*

Like any global symmetry, the BRST symmetry gives rise to a conserved current J^μ_B and charge Q_B, which has the following explicit form:

$$Q_B = \int d^3x \; J^0_B = \int d^3x \left[w^a D^{ab}_0 \eta^b - \dot{w}^a \eta^a + \frac{1}{2} i g \dot{\bar{\eta}}^a f^{abc} \eta^b \eta^c \right]$$

where $D_\mu^{ab} = \partial_\mu \delta^{ab} - g f^{abc} A_\mu^c$ is the covariant derivative in the adjoint representation, η^a, $\bar{\eta}^a$ are the ghost and anti-ghost fields, and w^a is the bosonic auxilliary field. It also turns out that these fields satisfy the following equations of motion:

$$\begin{aligned}
\partial^\mu (D_\mu \eta)^a &= 0 \\
(D^\mu \partial_\mu \bar{\eta})^a &= 0 \\
(D^\mu \partial_\mu w)^a &= igf^{abc}(\partial_\mu \bar{\eta}^b)(D^\mu \eta)^c
\end{aligned}$$

(a) Determine the equations of motion satisfied by η^a, $\bar{\eta}^a$ and w^a in the case where the gauge group is abelian.

(b) In this case show that Q_B takes the following form:

$$Q_B = i \int d^3p \left[a_\eta^\dagger(p) a_w(p) - a_w^\dagger(p) a_\eta(p) \right]$$

where $a_w^\dagger(p)/a_w(p)$ and $a_\eta^\dagger(p)/a_\eta(p)$ are the raising/lowering operators for the auxilliary and ghost fields respectively.

Hint. Perform a mode expansion of the ghost and auxilliary fields (with normalisation factors: $1/\sqrt{(2\pi)^3 2E_p}$), and apply normal ordering to the whole expression.

(c) Argue that in the abelian case the ghost fields completely decouple, i.e. that one can write: $\mathcal{V} = \mathcal{V}_{\text{phys}} \otimes \mathcal{V}_{FP}$, where $\mathcal{V}$ is the full space of states, $\mathcal{V}_{FP}$ is the Fock space of ghosts and anti-ghosts, and $\mathcal{V}_{\text{phys}}$ is the physical state space.

(d) In this formalism one defines a physical state to take the form: $|\text{phys}\rangle \equiv |f\rangle \otimes |0\rangle_{FP}$, where $|0\rangle_{FP} \in \mathcal{V}_{FP}$ is the ghost/anti-ghost vacuum, and $|f\rangle \in \mathcal{V}_{\text{phys}}$. Using the defining *BRST subsidiary condition* for a physical state:

$$Q_B |\text{phys}\rangle = 0$$

and the fact that the auxilliary field satisfies the equation of motion:

$$\partial^\mu A_\mu^a + \xi w^a = 0$$

show that in the abelian theory the BRST subsidiary condition reduces to the familiar *Gupta-Bleuler condition* encountered in the quantisation of the free electromagnetic field:

$$(\partial_\mu A^{\mu a})^{(+)} |\text{phys}\rangle = 0$$

where the + sign indicates only the positive frequency component of the field.

Hint. Apply the result of part (b) to $|phys\rangle$ and then use the equation of motion for w^a.