1/n

Goldstone theorem

Take a symmetry transformation

$$\phi_n(x) \to \phi_n(x) + i\epsilon \sum_m t_{nm} \phi_m(x) = \phi_n'(x)$$

with $\Gamma[\langle \phi_n(x) \rangle] = \Gamma[\langle \phi_n' \rangle]$

Slavnov-Taylor identities can be derived

$$\int d^4x \frac{\delta \Gamma}{\delta \phi_n} t_{nm} \langle \phi_m(x) \rangle = 0$$

Taking a second derivative

$$\int d^4x \frac{\delta^2 \Gamma[\langle \varphi \rangle]}{\delta \langle \phi_\ell(y) \rangle \, \delta \langle \phi_n(x) \rangle} t_{nm} \langle \phi_m(x) \rangle +$$

$$+ \frac{\delta \Gamma[\langle \varphi \rangle]}{\delta \langle \phi_\ell(y) \rangle} = 0$$

Consider physical systems where the source is zero

$$\frac{\delta \Gamma}{\delta \langle \phi \rangle} = 0$$

$$\rightsquigarrow \int d^4x \frac{\delta^2 \Gamma[\langle \phi \rangle]}{\delta \langle \phi_\ell(y) \rangle \, \delta \langle \phi_n(x) \rangle} t_{nm} \langle \phi_m(x) \rangle = 0.$$

$$\langle \phi(x) \rangle = \int \langle 0 | e^{ip\cdot x} \phi(0) e^{-i\hat{P}x} |0\rangle =$$

$$= \langle \phi(0) \rangle = \text{constant.} \quad \left(\begin{array}{c}\text{Important} \\ \text{assumption}\end{array} \right)$$

Then $\langle \phi_e(y) \rangle = \langle \phi_e \rangle$

and $\Gamma[\langle \phi \rangle] = -\int d^4x \; V_{\text{eff}}(\langle \phi \rangle)$

We get:

$$\boxed{\sum_{m,n} t_{nm} \frac{\partial^2 V(\langle \phi \rangle)}{\partial \langle \phi_e \rangle \partial \langle \phi_n \rangle} = 0}$$

Let's take:

$$\frac{\delta W[J]}{\delta J_n(x)} = \langle \phi_n(x) \rangle \rightsquigarrow$$

$$\rightsquigarrow \frac{\delta^2 W[J]}{\delta \langle \phi_n(y) \rangle \delta J_m(x)} = \delta(x-y)\, \delta_{nm}$$

$$\vdots$$

$$\rightsquigarrow \int d^4z \; \langle 0 | T\, \phi_k(x)\, \phi_n(z) | 0 \rangle \cdot$$

$$\cdot \frac{\delta^2 \Gamma}{\delta \langle \phi_n(z) \rangle \delta \langle \phi_m(y) \rangle} = -\delta^{(4)}(x-y)\, \delta_{km}$$

For $x=y=0$:

$$-\delta_{ne} \cdot \delta^{(4)}(0) = \int d^4z \, \frac{d^4p}{(2\pi)^4} \, e^{-ip(0-z)} \, \Delta_{ke}(p^2) \, \frac{\partial^2 V}{\partial \langle\phi_n\rangle \, \partial \langle\phi_k\rangle} \, d^4x$$

$$\Rightarrow \boxed{-\delta_{ne} \, \cancel{\delta^{(4)}(0)} = \Delta_{ke}(0^2) \, \frac{\partial^2 V}{\partial \langle\phi_n\rangle \, \partial \langle\phi_k\rangle} \, \cancel{\delta^{(4)}(0)}}$$

$$\sim \boxed{\frac{\partial^2 V}{\partial \langle\phi_n\rangle \, \partial \langle\phi_e\rangle} = \Delta_{ne}^{-1}(0)}$$

When is this satisfied?

Let's write:

$$\delta \langle\phi_n\rangle = i\epsilon \sum_m t_{nm} \langle\phi_m\rangle$$

$$\sim \boxed{\sum_n D_{en}^{-1}(0) \, \delta\langle\phi_n\rangle = 0} \quad (\text{Eq. A})$$

If symmetry leaves the vacuum unchanged, i.e. $\delta\langle\phi_n\rangle = 0$, then (Eq. A) is satisfied!

If the symmetry is broken; ϕ

$$\left(\Delta^{-1}_{\ell m}(0) \right) \begin{pmatrix} \\ \langle \phi_m \rangle \\ \\ \end{pmatrix} = 0 \begin{pmatrix} \langle \phi_m \rangle \\ \\ \end{pmatrix}$$

eigen-vector

with a zero eigenvalue:

$\exists$ linear combination of fields which has a zero inverse propagator as $p \to 0$.

so $\Delta^{-1}_{n\ell}(p) \to 0$ as $p \to 0$

and $\Delta_{n\ell}(p) \to \frac{1}{p^2}$ has a pole as $p^2 \to 0$.

We have proven that there is a massless particle in the spectrum of the theory for every $\delta \langle \phi_\ell \rangle = 0 = T_{\ell m} \langle \phi_m \rangle = 0$. generator ~~which~~ of the symmetry which changes the vacuum (broken generator).

$$\boxed{\begin{array}{l} \text{— Translation invariance !} \\ \text{— Positive norms !} \end{array}}$$

There is an alternative proof (for PESUB course)

* The massless-state is an one-particle state
* State is invariant under rotations: spin-0.
* It has the same "quantum numbers" as the conserved current corresponding to the broken generator.

Goldstone theorem is not valid for spontaneously broken local symmetries. The proof seems so powerfull that this has been hard to imagine. But it has its "gaps". Try to spot them!

- translation invariance
- positive norm of quantum states.

General broken global symmetries

Let's assume a SSB pattern.

$$\underbrace{G}_{\substack{\text{symmetry}\\\text{group of}\\ \Gamma[z(t)]}} \to \underbrace{H}_{\substack{\text{symmetry}\\\text{group of}\\\text{vacuum.}}} \quad , H \subset G.$$

$\mathcal{L}$ is invariant under assump. global. g

$$\varphi_n(x) \to \sum_m g_{nm}\,\varphi_m(x) \qquad \left(\frac{\partial g_{nm}}{\partial x^\mu} = 0\right)$$

Vacuum invariance under H means:

$$\sum_m h_{nm} \langle \varphi_m \rangle = \langle \varphi_n \rangle \qquad \forall h \in H.$$

The mass matrix of the theory has zero eigenvalues for the eigenvectors

$$\sum_m T^\alpha_{nm} \langle \varphi_m \rangle \equiv \delta \langle \varphi_n \rangle$$

(if T^α is a broken generator $\delta\langle\varphi_n\rangle \neq 0$)

Which are the Goldstone boson fields and which are not?

We shall prove that we can obtain all field combinations by performing a local group transformation on Goldstone-free field combinations:

$$\psi_n(x) = \sum_m \mathcal{T}^{\alpha}_{nm}(x)\, \tilde{\psi}_m(x)$$

→ Goldstone free.

Goldstone free combinations must be orthogonal to the vectors which constraint the "mass-matrix" to have zero eigenvalues:

$$\underline{T^{\alpha}_{nm}\langle\psi_m\rangle} \quad : \left(D^{-1}_{\ell n}\right)\cdot\left(T^{\alpha}_{nm}\langle\psi_a\rangle_m\right) = 0$$

Thus, it must be

$$\boxed{\tilde{\psi}_n(x)\cdot T^{\alpha}_{nm}\langle\psi_m\rangle \overset{!}{=} 0}$$

⇓

Includes "heavy" fields of theory.

Consider

$$V_\psi(g) = \psi_n\, g_{nm}\langle\psi_m\rangle \qquad g\in G$$

G ↓ in real orthogonal representation

$V_\psi(g)$: real function continuous bounded.

Let us ~~ass~~ find a $g = \gamma$ such that $V_{\phi(x)}(\gamma)$ is a maximum for every space-time point. Then, it must be stationary under a variation (small) of the ~~ga~~ group parameter

$$\boxed{\delta\gamma_{nm} = i \sum_a \epsilon^\alpha \gamma_{nl} T^\alpha_{lm}}$$

Thus

$$0 = \delta V_{\phi(x)}(\gamma) = i \sum_\alpha \epsilon^\alpha \sum_{nml} \phi_n(x) \gamma_{nl}(x) T^\alpha_{lm} \cdot \langle \phi_m \rangle$$

Recalling that we have chosen an orthogonal and real representation of the group, we have

$$\gamma_{nl} = [\gamma_{ln}]^{-1}$$

Then

$$0 = \sum_\alpha \epsilon^\alpha \sum_{lm} \left[\sum_n \gamma^{-1}_{ln}(x) \phi_n(x)\right] \cdot \left(\underbrace{T^\alpha_{lm} \langle \phi_m \rangle}\right)$$

$\rightsquigarrow$ The combinations

$$\tilde{\phi}_l = \sum_n \gamma^{-1}_{ln}(x) \phi_n(x)$$

are orthogonal to the $\sum_m T^\alpha_{lm} \langle \phi_m \rangle$ eigenvectors which constrain the ~~mass~~ matrix. They put no constrain on it and thus they are not Goldstones.

Let's rewrite the Lagrangian substituting

$$\psi(x) = \gamma(x)\,\tilde{\psi}(x)$$

↳ local transformation.

$\mathcal{L}[\gamma(x)\tilde{\psi}(x)] =$

$= \mathcal{L}[\gamma(x_0)\tilde{\psi}(x)]$ + function of derivatives of $\gamma(x)$, $\tilde{\varphi}(x)$

global invariance

$= \mathcal{L}[\tilde{\varphi}(x)]$ + [function of $\partial_\mu \gamma(x)$ ...]

This forbids terms lacking derivatives for Goldstone bosons

$$\cancel{\sim m_B^2 B(x) B(x)}$$

Also, at low energies, Goldstone interactions vanish:

$\partial_\mu \gamma \rightarrow \partial_\mu B(x) \Rightarrow p^\mu$ in Feynman rules.

as $p^\mu \rightarrow 0$ interaction vanish.

Spontaneous symmetry breaking of local symmetries

Let's repeat the same rewriting

$\varphi = \gamma^{\dagger}\tilde{\varphi}$ for a Lagrangian

which is invariant under a local gauge transformation. (Recall that $\boxed{\sum_{nm} \tilde{\varphi}_n \cdot T^a_{nm} \langle\varphi_m\rangle = 0}$

$\hookrightarrow$ Goldstone free.

Then $\mathcal{L}[\gamma(x)\cdot\tilde{\varphi}] = \mathcal{L}[\tilde{\varphi}]$

$\Downarrow$

Goldstone bosons dissappear from Lagrangian!

We have found an exception of the Goldstone theorem. The rewriting $\varphi = \gamma\cdot\tilde{\varphi}$

is equivalent to choosing a gauge fixing condition

$$\boxed{\tilde{\varphi}\cdot(T^a\langle\varphi\rangle) = 0}.$$

Lagrangians which are locally invariant under a continuous symmetry transformation require gauge bosons, for forming covariant derivatives. Let's look at the quadratic terms:

$$\mathcal{L} \ni \frac{1}{2}\sum_{n}\left(\partial_\mu \tilde{\varphi}_n - i g \sum_{m,a} T^{\alpha}_{nm} A^{\alpha}_{\mu} \tilde{\varphi}_n\right)^2$$

and shift:

$$\varphi_n = \langle \varphi_n \rangle + \varphi_n' = v_n + \varphi_n' .$$

Then:

$$\mathcal{L} \ni \frac{1}{2}\sum_{n}(\partial_\mu \varphi_n')^2 - \frac{1}{2}\sum_{a\beta} \mu^2_{a\beta} A_{a\mu} A_\beta^\mu$$

with

$$\mu^2_{aB} = -g^2 \sum_{nm} T^{\alpha}_{nm} T^{\beta}_{ne} \, v_e \, v_m$$

[The mass matrix of gauge bosons is proportional to coupling constants]

T^a_{nm} is imaginary for real group-representations

Then $\boxed{\mu^2_{\alpha\beta} \geq 0}$

Assume that there is a real linear combination of generators which does not break the symmetry:

$$\left(\sum_\alpha c_a T^\alpha\right) \cdot v = 0$$

$$\sim \sum_{\alpha, n, m} c_a T^\alpha_{nm} v_m = 0$$

Then

$$\sum_B c_B \mu^2_{aB} =$$

$$= -g^2 \sum_B c_B \sum_{nme} T^\alpha_{nm} T_{ne} v_m v_e =$$

$$= -g^2 \sum_n \underbrace{\left(\sum_{\alpha, m} c_\alpha T^\alpha_{nm} v_m\right)}_{0} v_e T_{ne} = 0$$

~~The~~ $\rightsquigarrow$ $\boxed{\text{Unbroken generator} \Rightarrow \Rightarrow \text{Massless gauge boson}}$

The opposite is also valid: Assume ~~an unbroken~~ a massless gauge boson

$$c^T M^2 c \geq 0$$

$$\rightsquigarrow 0 = \sum_{\alpha B} c_\alpha c_B \mu^2_{\alpha B} = \left[\left(\sum c_a T^\alpha_m\right) \cdot v_m\right]^2 = 0$$

$\Rightarrow$ UNBROKEN generator.

Let's now look at the propagator of gauge bosons.

$$\frac{1}{2}\sum_a \left(\partial^\alpha_\lambda A_\nu - \partial_\nu A^a_\lambda\right)^2 - \frac{1}{2}\sum_{aB} \mu^2_{\alpha B} A_{a\lambda} A_{B\nu}$$

$$= \frac{1}{2}\sum_{a,B} A^\nu_\alpha \cdot \Delta_{a\nu,B\lambda} A_{B\lambda} + \text{total-derivatives}$$

with $\Delta_{a\mu,B\nu} = \delta_{aB}\left[g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right\} - \mu^2 g_{\mu\nu}\right]$

In momentum space:

$$\Delta^{\mu\nu,\alpha B}(k) = \frac{\delta_{\alpha B}}{k^2-\mu^2}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}\right]$$

At $k \to \infty$ $\Delta^{\mu\nu,\alpha B} \to$ constant.
~~Th~~ This spoils renormalizability arguments.

In massless gauge theory after gauge-fixing:

$$\Delta(k) \sim \frac{1}{k^2}\left[-g_{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right]$$