Problem Set 1: Scattering amplitudes in gauge theories

Discussion on Wednesday 26.02 12:45-14:30, HIT H 51
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Exercise 1 – Manipulating Spinor Indices

The ϵ symbols are used to raise and lower Weyl indices according to $\bar{\xi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}^{\dot{\beta}}$ and $\chi^{\alpha} = \epsilon^{\alpha\beta}\,\chi_{\beta}$. We have

$$\epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = \epsilon^{21} = \epsilon^{\dot{2}\dot{1}} = 1\,, \qquad \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = \epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = -1\,.$$

The sigma matrix is defined by $(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} = (\mathbf{1}, -\vec{\sigma})$. Moreover we have $\sigma^{\mu}_{\alpha\dot{\alpha}} := \epsilon_{\alpha\beta}\,\epsilon_{\dot{\alpha}\dot{\beta}}\,\bar{\sigma}^{\mu\,\dot{\beta}\beta}$. Prove the relations

$$\sigma^{\mu}_{\alpha\dot{\alpha}} = (\mathbf{1},\vec{\sigma})\,, \qquad \sigma_{\mu\,\alpha\dot{\alpha}} = (\mathbf{1}, -\,\vec{\sigma})\,,$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\,\sigma_{\mu\,\beta\dot{\beta}} = 2\,\epsilon_{\alpha\beta}\,\epsilon_{\dot{\alpha}\dot{\beta}}\,, \qquad \epsilon^{\alpha\beta}\,\epsilon^{\dot{\alpha}\dot{\beta}}\,\sigma^{\mu}_{\alpha\dot{\alpha}}\,\sigma^{\nu}_{\beta\dot{\beta}} = 2\,\eta^{\mu\nu}\,.$$

Exercise 2 – Massless Dirac equation and Weyl Spinors

Consider the (standard) representation of the Dirac matrices

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}\,, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\,, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}\,.$$

a) Show that the solutions of the massless Dirac equation $\gamma^{\mu}\,k_{\mu}\psi = 0$ may be chosen as

$$u_+(k) = v_-(k) = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{k^+} \\ \sqrt{k^-}\,e^{i\phi(k)} \\ \sqrt{k^+} \\ \sqrt{k^-}\,e^{i\phi(k)} \end{pmatrix}\,, \quad u_-(k) = v_+(k) = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{k^-}\,e^{-i\phi(k)} \\ -\sqrt{k^+} \\ -\sqrt{k^-}\,e^{-i\phi(k)} \\ \sqrt{k^+} \end{pmatrix}$$

where

$$e^{\pm i\phi(k)} := \frac{k^1 \pm ik^2}{\sqrt{k^+\,k^-}} \qquad k^{\pm} := k^0 \pm k^3\,,$$

and show that the spinors $u_{\pm}(k)$ and $v_{\pm}(k)$ obey the helicity relations

$$P_{\pm} := \frac{1}{2}\,(\mathbf{1} \pm \gamma^5)\,, \qquad P_{\pm}u_{\pm} = u_{\pm}\,, \quad P_{\pm}u_{\mp} = 0\,, \quad P_{\pm}v_{\pm} = 0\,, \quad P_{\pm}v_{\mp} = v_{\mp}\,.$$

b) What helicity relations hold for the conjugate expressions $\bar{u}_{\pm}(k)$ and $\bar{v}_{\pm}(k)$, where of course $\bar{\psi} := \psi^{\dagger}\gamma^0$?

c) Now consider the unitary transformation

$$\psi \to U\,\psi\;, \qquad \gamma^\mu \to U\,\gamma^\mu\,U^\dagger\;,$$

using $U = \frac{1}{\sqrt{2}}\,(1 - i\gamma^1\,\gamma^2\,\gamma^3)$ to the chiral representation of the Dirac matrices:

$$\gamma^0_{ch} = \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix}\;, \qquad \gamma^i_{ch} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\;, \qquad i = 1{,}2{,}3\,.$$

Determine γ^5 and the spinors $u_\pm(k)$ and $v_\pm(k)$ in this chiral basis!