Introduction to Spin Chains
and
Supersymmetric Gauge Theories

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Lecture 1

Introduction and Motivation

Before formally introducing supersymmetric gauge theories and spin chains, I will give a brief overview about the two subjects and the relations between them, to provide the student with a sketch of the big picture.

Gauge theories are the foundation of our understanding of nature. Of the fundamental interactions, the electroweak force and QCD (strong interaction) are described by quantum gauge theories. Understanding gauge theories as well as possible is a top priority. Despite decades of research, there are still open problems remaining, in particular regarding their non-perturbative behavior and confinement. One way of rendering quantum gauge theories more tractable is to introduce supersymmetry.

Supersymmetry is a symmetry relating bosons and fermions. Each particle has a superpartner of the opposite statistics. We speak about *extended* supersymmetry when we have more than one set of supersymmetry generators and superpartners (*e.g.* $\mathcal{N} = 2$, $\mathcal{N} = 4$ supersymmetry). Supersymmetry has not yet been observed in nature. It must necessarily be broken, as otherwise particles and their superpartners would have the same mass, which is clearly not the case. Even so, the allowed regions for the superpartner masses that have not yet been excluded by experiment are shrinking. Here, we will however not worry ourselves with phenomenological concerns. We will be using supersymmetry as a tool to gain insights into gauge theories, as a kind of *laboratory* for studying them. Supersymmetry constrains a theory and makes it well-behaved. It has a number of desirable mathematical properties, such as *e.g.* non-renormalisation theorems and protection of certain quantities from quantum corrections. The more supersymmetry a theory has, the more constrained it is, but a the same time, the less realistic it is from a phenomenological point of view (*e.g.* $\mathcal{N} = 4$ super Yang–Mills theory).

In recent years, $\mathcal{N} = 2$ gauge theories have been a focus of interest. Seiberg and Witten (1994) showed that $\mathcal{N} = 2$ SYM theory can be solved completely at the quantum level. It is possible to construct an exact low energy Lagrangian and the exact spectrum of BPS states. It displays moreover a strong/weak duality and has a rich algebraic structure surviving quantum corrections.

In the following, we will be particularly interested in *deformations* of supersymmetric gauge theories that preserve some of the supersymmetry and in particular preserve its useful properties. There will be two types of deformation of relevance:

- mass deformations (*e.g.* twisted mass deformations in 2D).
- Ω–type deformations.

In the Ω–deformation, deformation parameters ϵ_i are introduced which break Poincaré symmetry. It was introduced by Nekrasov (2004) as a calculational device for a localization calculation of the instanton sum of $\mathcal{N} = 2$ SYM. However, it is also interesting to study Ω–deformed gauge theories in their own right.

We will see later on that these deformed gauge theories are intimately connected with integrable systems.

Integrable models are exactly solvable models of many-body systems in statistical mechanics or solid state physics. These models are inherently discrete (lattice models). With some exceptions, most known integrable models are one-dimensional.

In 1931, Hans Bethe solved the $\text{XXX}_{1/2}$ or Heisenberg spin chain. His ansatz can be generalized to many more systems and is the basis of the field of integrable systems. Colloquially, people often equate "Bethe-solvable" with "integrable", but the set of integrable models is marginally bigger.

The example we will be mostly concerned with is the one of a *1d ferromagnet*. Here, one studies a linear chain of L identical atoms with only next-neighbor interactions. Each atom has one electron in an outer shell (all other shells being complete). These electrons can either be in the state of spin up ($\uparrow$) or down ($\downarrow$). At first order, the Coulomb- and magnetic interactions result in the *exchange interaction* in which the states of neighboring spins are interchanged:

$$\uparrow\downarrow \quad \leftrightarrow \quad \downarrow\uparrow \tag{1.1}$$

In a given spin configuration of a spin chain, interactions can happen at all the anti-parallel pairs. For simplicity, we will be studying the periodic chain. Bethe posed himself the question of finding the spectrum and energy eigenfunctions of this spin chain. We will study his ansatz in detail in the next chapter. His method is a little gem and studying it is likely to lift the morale of any theoretical physicist!

There are many generalizations to this simplest of all spin chains which can still be solved by versions of Bethe's ansatz:

- different *boundary conditions*: periodic, anti-periodic, open, kink, ...
- *anisotropic* models: XXY chain, where the z-direction is singled out by a magnetic field in this directions, XYZ model.
- different choice of *symmetry group*. The spin $1/2$ spin chain corresponds to $SU(2)$, but any Lie group or even supergroup can be chosen instead.
- for the *rank* of the symmetry group $r > 1$, there are more particle species on the chain, *e.g.* $\uparrow$, $\downarrow$, $\circ$ (hole) of the tJ–model, where $\uparrow$, $\downarrow$ are fermionic while $\circ$ is bosonic.
- each site of the spin chain can carry a different *representation* of the symmetry group.
- on each site another parameter, the so-called *inhomogeneity* can be turned on.

The relations between integrable models and supersymmetric gauge theories are a very interesting subject and active research topic. There are several examples of these connections, *e.g.*

- *2d gauge/Bethe correspondence*: $\mathcal{N} = (2, 2)$ gauge theories in 2d are related to Bethe-solvable spin chains.
- *4d gauge/Bethe correspondence*: Ω–deformed $\mathcal{N} = 2$ supersymmetric gauge theories are related to quantum integrable models.

- *Alday–Gaiotto-Tachikawa (AGT) correspondence*: Ω–deformed super Yang–Mills theory in 4d is related to Liouville theory.

In these lectures, we will concentrate on the 2d gauge/Bethe correspondence (Nekrasov–Shatashvili). We will match the parameters of a spin chain to those of $\mathcal{N} = (2, 2)$ supersymmetric gauge theories. It will turn out that the full (Bethe) spectrum of the spin chain corresponds one-to-one to the supersymmetric ground states of the corresponding gauge theories. In order to understand this correspondence, we first need to introduce the following concepts in the course of the coming lectures:

Spin Chain:

- parameters of a general spin chain
- Bethe ansatz equations
- Yang–Yang counting function

Supersymmetric gauge theories:

- $\mathcal{N} = (2, 2)$ gauge theories
- twisted mass deformation
- low energy effective action, in particular the effective twisted superpotential
- equation for the ground states

Lecture 2

Spin Chains and the Bethe Ansatz

As discussed in the introduction, we want to find the energy eigenfunctions and eigenvalues of a 1d magnet. Consider a closed chain of identical atoms with each one external electron which can be in the state of spin up or down and only next-neighbor interactions — the $\mathrm{XXX}_{1/2}$ spin chain. Its Hamiltonian (Heisenberg 1926) is given by

$$H = -J \sum_{n=1}^{L} P_{n,n+1}, \tag{2.1}$$

where J is the exchange integral[1] and $P_{n,n+1}$ is the permutation operator of states at positions $n,\ n+1$. Let us write down the spin operator at position n on the spin chain:

$$\vec{S}_n = (S_n^x, S_n^y, S_n^z) = \tfrac{1}{2}\vec{\sigma}_n, \tag{2.2}$$

where σ_n are the Pauli matrices for spin $1/2$. In terms of the spin operators, the permutation operator is given by

$$P_{n,n+1} = -\tfrac{1}{2}(1 + \vec{\sigma}_n \vec{\sigma}_{n+1}). \tag{2.3}$$

In terms of the spin operators, the Hamiltonian (2.1) becomes

$$\begin{aligned} H &= -J \sum_{n=1}^{L} \vec{S}_n \vec{S}_{n+1} \\ &= -J \sum_{n=1}^{L} \tfrac{1}{2}\left(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+\right) + S_n^z S_{n+1}^z, \end{aligned} \tag{2.4}$$

where $S^\pm = S_n^x \pm i S_n^y$ are the *spin flip operators*. The term in parentheses corresponds to the *exchange interaction* which exchanges neighboring spin states.

The spin flip operators act as follows on the spins:

$$\begin{aligned} S_k^+ |\ldots \uparrow \ldots\rangle &= 0, & S_k^+ |\ldots \downarrow \ldots\rangle &= |\ldots \uparrow \ldots\rangle, \\ S_k^- |\ldots \uparrow \ldots\rangle &= |\ldots \downarrow \ldots\rangle, & S_k^- |\ldots \downarrow \ldots\rangle &= 0, \\ S_k^z |\ldots \uparrow \ldots\rangle &= \tfrac{1}{2}|\ldots \uparrow \ldots\rangle, & S_k^z |\ldots \downarrow \ldots\rangle &= -\tfrac{1}{2}|\ldots \downarrow \ldots\rangle. \end{aligned} \tag{2.5}$$

The spin operators have the following commutation relations:

$$[S_n^z, S_{n'}^\pm] = \pm S_n^\pm \delta_{nn'}, \quad [S_n^+, S_{n'}^-] = 2S_n^z \delta_{nn'}. \tag{2.6}$$

For the *closed* chain, the sites n and $n+L$ are identified:

$$\vec{S}_{L+1} = \vec{S}_1. \tag{2.7}$$

[1] $-J > 0$: ferromagnet, spins tend to align, $-J < 0$: anti-ferromagnet, spins tend to be anti-parallel.

So far, we have discussed the *isotropic* spin chain. In the anisotropic case, a magnetic field is turned on in the z-direction:

$$H_\Delta = -J \sum_{n=1}^{L} S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta (S_n^z S_{n+1}^z - \tfrac{1}{4}). \tag{2.8}$$

This is the XXZ spin chain. Δ is the anisotropy parameter, where $\Delta = 1$ is the isotropic case. The most general model in this respect is the XYZ spin chain:

$$H_{\Delta,\Gamma} = -\sum_{n=1}^{L} J_x S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + S_n^z S_{n+1}^z. \tag{2.9}$$

It has two anisotropy parameters Δ, Γ which fulfill the ratios

$$J_x : J_y : J_z = 1 - \Gamma : 1 + \Gamma : \Delta. \tag{2.10}$$

In the following, we will however concentrate on the isotropic case.

Let us define the *ferromagnetic reference state*

$$|\uparrow\uparrow \ldots \uparrow\uparrow\rangle = |\Omega\rangle. \tag{2.11}$$

H acts on a Hilbert space of dimension 2^L, given that each site on the chain can be in one of two states, which is spanned by the orthogonal basis vectors

$$|\Omega(n_1, \ldots, n_N)\rangle = S_{n_1}^- \ldots S_{n_N}^- |\Omega\rangle, \tag{2.12}$$

which are vectors with N down spins ($0 \leq N \leq L$) in the positions $n_1, \ldots, n_N$, where we always take $1 \leq n_1 < n_2 < \cdots < n_N \leq L$.

In order to diagonalize the Heisenberg model, two symmetries will be of essential importance:

- the conservation of the z–component of the total spin,

$$[H, S^z] = 0, \quad S^z = \sum_{n=1}^{L} S_n^z. \tag{2.13}$$

  This remains also true for the XXZ spin chain Hamiltonian H_Δ.

- the translational symmetry, *i.e.* the invariance of H with respect to discrete translations by any number of lattice spacings. This symmetry results from the periodic boundary conditions we have imposed.

As the exchange interaction only moves down spins around, the number of down spins in a basis vector is not changed by the action of H. Acting with H on $|\Omega(n_1, \ldots, n_N)\rangle$ thus yields a linear combination of basis vectors with N down spins. It is therefore possible to *block-diagonalize* H by sorting the basis vectors by the quantum number $S^z = L/2 - N$.

Let us start by considering the subsector with $N = 0$. It contains only one single basis vector, namely $|\Omega\rangle$, which is an eigenvector of H as there are no antiparallel spins for the exchange interaction to act on:

$$H|\Omega\rangle = E_0|\Omega\rangle, \quad E_0 = -J\tfrac{L}{4}. \tag{2.14}$$

Next we consider the sector with $N = 1$. As the down spin can be in each of the lattice sites, this subspace is spanned by

$$|\Omega(n)\rangle = S_n^- |\Omega\rangle. \tag{2.15}$$

In order to diagonalize this block, we must invoke the translational symmetry. We can construct translationally invariant basis vectors as follows:

$$|\psi\rangle = \frac{1}{\sqrt{L}} \sum_{n=1}^{L} e^{ikn} |\Omega(n)\rangle, \quad k = \frac{2\pi m}{L},\ m = 0, 1, \ldots, L-1. \tag{2.16}$$

The $|\psi\rangle$ with wave number k are eigenvectors of the translation operator with eigenvalue e^{ik} and eigenvectors of H with eigenvalues

$$E = -J(\tfrac{L}{4} - 1 - \cos k), \tag{2.17}$$

or in terms of E_0,

$$E - E_0 = J(1 - \cos k). \tag{2.18}$$

The $|\psi\rangle$ are so-called *magnon* excitations: the ferromagnetic ground state is periodically perturbed by a *spin wave* with wave length $2\pi/k$.

So far, we have block-diagonalized H and diagonalized the sectors $N = 0, 1$ by symmetry considerations alone. The invariant subspaces with $2 \leq N \leq L/2$ however are not completely diagonalized by the translationally invariant basis.

In order to remedy this situation, we will now study Bethe's ansatz, again for the case $N = 1$.

Bethe ansatz for the one-magnon sector. We can write the eigenvectors of H in this sector as

$$|\psi\rangle = \sum_{n=1}^{L} a(n) |\Omega(n)\rangle. \tag{2.19}$$

Plugging this into the eigenvalue equation results in a set of conditions for $a(n)$:

$$2\left[E + \frac{JL}{4}\right] a(n) = J\left[2a(n) - a(n-1) - a(n+1)\right], \quad n = 1, 2, \ldots, L. \tag{2.20}$$

On top of this, we have the periodic boundary conditions

$$a(n + L) = a(n). \tag{2.21}$$

The L linearly independent solutions to the difference equation Eq. (2.20) are given by

$$a(n) = e^{ikn}, \quad k = \frac{2\pi}{L} m,\ m = 0, 1, \ldots, L-1. \tag{2.22}$$

Little surprisingly, these are the same solutions we had found before. But now we can apply the same procedure to the case $N = 2$.

Bethe ansatz for the two-magnon sector. This invariant subspace has dimension $L(L-1)/2$. We want to determine $a(n_1, n_2)$ for the eigenstates of the form

$$|\psi\rangle = \sum_{1 \leq n_1 < n_2 \leq L} a(n_1, n_2) |\Omega(n_1, n_2)\rangle. \tag{2.23}$$

Bethe's preliminary ansatz is given by

$$a(n_1, n_2) = A\, e^{i(k_1 n_1 + k_2 n_2)} + A'\, e^{i(k_1 n_2 + k_2 n_1)}. \tag{2.24}$$

The first term is called the *direct term* and represents an incoming wave, while the second term is called the *exchange term* and represents an outgoing wave. Indeed, the expression looks like the superposition of two magnons, however, the flipped spins must always be in different lattice sites. Asymptotically, we can only have the direct and exchange terms for two magnons. Bethe's ansatz postulates that this asymptotic form remains true in general.

Let us now plug the eigenstates $|\psi\rangle$ of the form given in Eq. (2.24) into the eigenvalue equation. There are two cases to consider separately, namely the two down spins not being adjacent, and the two down spins being adjacent:

$$\begin{aligned}2(E-E_0)\,a(n_1,n_2) &= J\,[4\,a(n_1,n_2)+a(n_1-1,n_2)-a(n_1+1,n_2)\\ &\quad -a(n_1,n_2-1)-a(n_1,n_2+1)]\,,\quad n_2>n_1+1\,,\end{aligned} \tag{2.25}$$

$$2(E-E_0)\,a(n_1,n_2) = J\,[2a(n_1,n_2)-a(n_1-1,n_2)-a(n_1,n_2+1)]\,,\quad n_2=n_1+1. \tag{2.26}$$

Equations (2.25) are satisfied by $a(n_1,n_2)$ of the form Eq. (2.24) with arbitrary A, A', k_1, k_2 for both $n_2>n_1+1$ and $n_2=n_1+1$ if the energies fulfill

$$E-E_0 = J\sum_{j=1,2}(1-\cos k_j). \tag{2.27}$$

Equation (2.26) on the other hand is not automatically satisfied. Subtracting equation (2.26) from equation (2.25) for the case $n_2=n_1+1$ leads to N conditions, known as the *meeting conditions*:

$$2\,a(n_1,n_1+1) = a(n_1,n_1)+a(n_1+1,n_1+1). \tag{2.28}$$

Clearly, the expressions $a(n_1,n_1)$ have no physical meaning, as the two down spins cannot be at the same site, but are defined formally by the ansatz Eq. (2.24). Thus the $a(n_1,n_2)$ solve Eq. (2.25), (2.26) if they have the form Eq. (2.24) and fulfill Eq. (2.28). Plugging Eq. (2.24) into Eq. (2.28) and taking the ratio, we arrive at

$$\frac{A}{A'} =: e^{i\theta} = -\frac{e^{i(k_1+k_2)}+1-2\,e^{ik_1}}{e^{i(k_1+k_2)}+1-2\,e^{ik_2}}. \tag{2.29}$$

We see thus that as a result of the magnon interaction, we get an extra phase factor in the Bethe ansatz Eq. (2.24):

$$a(n_1,n_2) = e^{i(k_1n_1+k_2n_2+\frac{1}{2}\theta_{12})}+e^{i(k_1n_2+k_2n_1+\frac{1}{2}\theta_{21})}, \tag{2.30}$$

where $\theta_{12}=-\theta_{21}=\theta$, or written in the real form,

$$2\cot\theta/2 = \cot k_1/2-\cot k_2/2. \tag{2.31}$$

k_1, k_2 are the momenta of the Bethe ansatz wave function. The translational invariance of $|\psi\rangle$,

$$a(n_1,n_2) = a(n_2,n_1+L) \tag{2.32}$$

is satisfied if

$$e^{ik_1L}=e^{i\theta},\quad e^{ik_2L}=e^{-i\theta}, \tag{2.33}$$

which, after taking the logarithm, is equivalent to

$$L\,k_1 = 2\pi\tilde{\lambda}_1+\theta,\quad L\,k_2 = 2\pi\tilde{\lambda}_2+\theta, \tag{2.34}$$

where $\tilde{\lambda}_i\in\{0,1,\ldots,L-1\}$ are the *Bethe quantum numbers* which fulfill

$$k = k_1+k_2 = \tfrac{2\pi}{L}(\tilde{\lambda}_1+\tilde{\lambda}_2). \tag{2.35}$$

We have seen that the expression for the energies, Eq. (2.27), is reminiscent of two superimposed magnons. The magnon interaction is reflected in the phase shift θ and the deviation of the momenta k_1, k_2 from the one-magnon wave numbers. We will see that the magnons either *scatter* off each other or form *bound states*. In the following lectures, we will be mostly interested in the form of the Bethe equations themselves, and not so much in their explicit solutions. But before treating the general N magnon case, we will nonetheless quickly review the properties of the Bethe eigenstates for $N = 2$.

We need to identify all pairs $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ that satisfy the Bethe Equations (2.31) and (2.34). Allowed pairs are restricted to $0 \le \tilde{\lambda}_1 \le \tilde{\lambda}_2 \le L - 1$. Switching $\tilde{\lambda}_1$ with $\tilde{\lambda}_2$ interchanges k_1 and k_2 and leads to the same solution. $L(L+1)/2$ pairs meet the restriction, however only $L(L-1)/2$ of them produce solutions, which corresponds to the size of the Hilbert space. There are three distinct classes of solutions:

1. *One of the Bethe quantum numbers is zero:* $\tilde{\lambda}_1 = 0$, $\tilde{\lambda}_2 = 0, 1, \ldots, L-1$. There exists a *real* solution for all N combinations $k_1 = 0$, $k_2 = 2\pi\tilde{\lambda}_2/L$, $\theta = 0$. These solutions have the same dispersion relation as the one-magnon states in the subspace $N = 1$.

2. $\tilde{\lambda}_1, \tilde{\lambda}_2 \neq 0$, $\tilde{\lambda}_2 - \tilde{\lambda}_1 \ge 2$. There are $L(L-5)/2 + 3$ such pairs and each gives a solution with real k_1, k_2. These solutions represent nearly free superpositions of two one-magnon states.

3. $\tilde{\lambda}_1, \tilde{\lambda}_2 \neq 0$, $\tilde{\lambda}_1, \tilde{\lambda}_2$ *are either equal or differing by unity.* There are $2N - 3$ such pairs, but only $N - 3$ yield solutions. Most are complex, $k_1 := k/2 + iv$, $k_2 := k/2 - iv$, $\theta := \phi + i\chi$. These solutions correspond to two-magnon bound states. They exhibit an enhanced probability that the two flipped spins are on neighboring sites.

The number of solutions adds up to the dimension of the Hilbert space. The first and second class of solutions correspond to two-magnon scattering states.

Bethe ansatz for the N–magnon sector. We are finally ready to tackle the general case with an unrestricted number $N \le L$ of down spins. This subspace has dimension $L!/((L-N)!N!)$. The eigenstates have the form

$$|\psi\rangle = \sum_{1 \le n_1 < \cdots < n_N \le L} a(n_1, \ldots, n_N) |\Omega(n_1, \ldots, n_N)\rangle. \tag{2.36}$$

Here, we have N momenta k_j and one phase angle $\theta_{ij} = -\theta_{ji}$ for each pairs (k_i, k_j). The *Bethe ansatz* now has the form

$$a(n_1, \ldots, n_N) = \sum_{\mathcal{P} \in S_N} \exp\left(i \sum_{j=1}^{N} k_{p(j)} n_j + \frac{i}{2} \sum_{i<j} \theta_{p(i)p(j)} \right), \tag{2.37}$$

where $\mathcal{P} \in S_N$ are the $N!$ permutations of $\{1, 2, \ldots, N\}$. From the eigenvalue equation, we again get the two kinds of difference equations (the first for no adjacent down spins, the second for one pair of adjacent down spins),

$$2[E-E_0]\,a(n_1,\dots,n_N) = J\sum_{i=1}^{N}\sum_{\sigma=\pm1}[a(n_1,\dots,n_N) - a(n_1,\dots,n_{i+\sigma},\dots,n_N)], \qquad \text{if } n_{j+1} > n_j+1,\ j=1,\dots,N, \tag{2.38}$$

$$\begin{aligned}2[E-E_0]\,a(n_1,\dots,n_N) = {} & J\sum_{i\neq j_\alpha, j_\alpha+1}^{N}\sum_{\sigma=\pm1}[a(n_1,\dots,n_N) - a(n_1,\dots,n_{i+\sigma},\dots,n_N)] \\ & + J\sum_{\alpha}[2a(n_1,\dots,n_N) - a(n_1,\dots,n_{j_\alpha}-1,n_{j_\alpha+1},\dots,n_N) \\ & - a(n_1,\dots,n_{j_\alpha},n_{j_\alpha+1}+1,\dots,n_N)], \\ & \text{if } n_{j_\alpha}+1 = n_{j_\alpha+1},\ n_{j+1} > n_j+1,\ j\neq j_\alpha.\end{aligned} \tag{2.39}$$

The coefficients $a(n_1,\dots,n_N)$ are solutions of Equations (2.38), (2.39) for the energy

$$E-E_0 = J\sum_{j=1}^{N}(1-\cos k_j) \tag{2.40}$$

if they have the form Eq. (2.37) and fulfill the N meeting conditions

$$\begin{aligned}2\,a(n_1,\dots,n_{j_\alpha},n_{j_\alpha}+1,\dots,n_N) = {} & a(n_1,\dots,n_{j_\alpha},n_{j_\alpha},\dots,n_N) \\ & + a(n_1,\dots,n_{j_\alpha}+1,n_{j_\alpha}+1,\dots,n_N),\end{aligned} \tag{2.41}$$

for $\alpha = 1,\dots,N$. This relates the phase angles to the (not yet determined) k_j:

$$e^{i\theta_{ij}} = -\frac{e^{i(k_i+k_j)+1-2\,e^{ik_i}}}{e^{i(k_i+k_j)+1-2\,e^{ik_j}}}, \tag{2.42}$$

or, in the real form

$$2\,\cot\theta_{ij}/2 = \cot k_i/2 - \cot k_j/2, \quad i,j = 1,\dots,N. \tag{2.43}$$

Translational invariance, respectively the periodicity condition

$$a(n_1,\dots,n_N) = a(n_2,\dots,n_N,n_1+L) \tag{2.44}$$

gives rise to

$$\sum_{j=1}^{N} k_{p(j)}n_j + \frac{1}{2}\sum_{i<j}\theta_{p(i)p(j)} = \frac{1}{2}\sum_{i<j}\theta_{p'(i)p'(j)} - 2\pi\lambda_{p'(N)} + \sum_{j=2}^{N} k_{p'(j-1)}n_j + k_{p'(N)}(n_1+L), \tag{2.45}$$

where $p'(i-1) = p(i)$, $i = 1,2,\dots,N$ and $p'(N) = p(1)$. All terms not involving $p'(N) = p(1)$ cancel, we are therefore left with N relations

$$L\,k_i = 2\pi\lambda_i + \sum_{j\neq i}\theta_{ij}, \tag{2.46}$$

with $i = 1,\dots,N$ and $\lambda_i \in \{0,1,\dots,L-1\}$. We need to again find sets of Bethe quantum numbers $(\lambda_1,\dots,\lambda_N)$ which lead to solutions of the Bethe equations (2.42), (2.46). Each solution represents an eigenvector of the form Eq. (2.37) with energy (2.40) and wave number

$$k = \frac{2\pi}{L}\sum_{i=1}^{L}\lambda_i. \tag{2.47}$$

Similarly to the two magnon case, bound state solutions appear, this time also with three or more magnons.

In order to find a clear interpretation of the Bethe ansatz, let us rewrite the N–particle ansatz Eq. (2.37) as follows:

$$\begin{aligned} a(n_1,\ldots,n_N) &= \sum_{\mathcal{P}\in S_N} \exp\left(\frac{i}{2}\sum_{i<j}\theta_{p(i)p(j)}\right)\exp\left(i\sum_{j=1}^{N}k_{p(j)}n_j\right) \\ &= \sum_{\mathcal{P}\in S_N} A(k_{p(1)},\ldots,k_{p(N)})\exp\left(i\sum_{j=1}^{N}k_{p(j)}n_j\right). \end{aligned} \tag{2.48}$$

The coefficient $A(k_{p(1)},\ldots,k_{p(N)})$ factorizes into pair interactions:

$$A(k_{p(1)},\ldots,k_{p(N)}) = \prod_{1\leq i<j\leq N} e^{\frac{i}{2}\theta_{ij}}. \tag{2.49}$$

We have seen that the two-body interactions are not free, they have a non-trivial scattering matrix. The many-body collisions factorize, which means they happen as a sequence of two magnon collisions. All Bethe-solvable systems are thus *two-body reducible*. This property has to do with the fact that a spin chain is one-dimensional, so only neighboring down spins can interact directly. There are in fact very few two-dimensional systems that are exactly solvable.

To conclude this part, we will re-write the Bethe equations to give them a form which is more commonly used in the literature and which we will need in the last part of this lecture series. First, we introduce new variables, the so-called *rapidities* λ_i:

$$e^{ik_j} = \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}. \tag{2.50}$$

Plugging them into the periodicity condition, we get

$$\left(\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}\right)^L = \prod_{j\neq i}^{N}\frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}, \quad i = 1,\ldots,N. \tag{2.51}$$

This Bethe equation encodes the periodic boundary condition. In can be generalized to boundary conditions with a twist ϑ,

$$\vec{S}_{L+1} = e^{\frac{i}{2}\vartheta\sigma_z}\vec{S}_1 e^{-\frac{i}{2}\vartheta\sigma_z}: \tag{2.52}$$

$$\left(\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}\right)^L = e^{i\vartheta}\prod_{j\neq i}^{N}\frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}, \quad i = 1,\ldots,N. \tag{2.53}$$

The Bethe ansatz as it was presented in this lecture follows Bethe's original treatment and is referred to as the *coordinate Bethe ansatz*. It has the advantage that its physics is intuitively very clear. It can be generalized to the XXZ spin chain, but not much beyond that. The so-called *algebraic* Bethe ansatz is mathematically more elegant and much more powerful. It uses concepts such as the Yang–Baxter equations, the Lax operator and the R–matrix and relies heavily on the machinery of group theory. This goes beyond the scope of the present lecture series.

As a last remark, we make the non-trivial observation that the Bethe equations (2.53) describe the critical points of a potential, the so-called *Yang–Yang counting function* Y. We can rewrite the Bethe equations as

$$e^{2\pi i \varpi(\lambda)} = 1. \tag{2.54}$$

The one-form $\varpi = \sum_{j=1}^{N} \varpi_j(\lambda) \mathrm{d}\lambda_j$ is closed and $\varpi = \mathrm{d}Y$, with

$$Y(\lambda) = \frac{L}{2\pi} \sum_{i=1}^{N} \hat{x}(2\lambda_i) - \frac{1}{2\pi} \sum_{i,j=1}^{N} \hat{x}(\lambda_i - \lambda_j) + \sum_{j=1}^{N} \lambda_j \left(n_j - \frac{\vartheta}{2\pi} \right), \tag{2.55}$$

$$\hat{x}(\lambda) = \lambda \tfrac{i}{2} \left(\log(1 - \tfrac{i}{\lambda}) - \log(1 + \tfrac{i}{\lambda}) \right) + \tfrac{1}{2} \log(1 + \lambda^2), \tag{2.56}$$

where the n_i are integers. The Bethe equations thus ultimately take the form

$$e^{2\pi \mathrm{d}Y(\lambda)} = 1. \tag{2.57}$$

Literature. This lecture follows largely [1], which itself follows Bethe's original work [2]. The first chapter of [3] is also very useful as an introduction.

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Lecture 3

Supersymmetric Gauge Theories in 2D

This part follows closely sections 12.1, 12.2, 15.2 and 15.5 of the book *Mirror Symmetry* by K. Hori et al [1].

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Lecture 4

Relations between Spin Chains and Supersymmetric Gauge Theories

As we have seen in the introduction, there are a number of interesting relations between supersymmetric gauge theories and integrable models. We will here concentrate on the *gauge/Bethe correspondence* in 2d of Nekrasov and Shatashvili. This correspondence maps the parameters of *any* Bethe solvable spin chain to a set of $\mathcal{N} = (2,2)$ supersymmetric gauge theories in 2d. We will see that the N–magnon Bethe spectrum is identified with the supersymmetric ground states of a rank N gauge theory. Having learned in the previous lectures how to obtain the Bethe equations that govern the spectrum of a spin chain and how to obtain the ground states of the low energy effective gauge theory, we are ready to tackle at least the simplest example of this correspondence:

Example: Spin $1/2$ Heisenberg model - $XXX_{1/2}$ spin chain. Recall the Bethe ansatz equation for the N–magnon sector, Eq. (2.51):

$$\left(\frac{\lambda_i + \frac{i}{2}}{\lambda_i - \frac{i}{2}}\right)^L = \prod_{\substack{j=1\\ j\neq i}}^{N} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}, \quad i = 1, \ldots, N\,. \tag{4.1}$$

We have seen that it is expressed equivalently by

$$e^{2\pi \mathrm{d}Y(\lambda)} = 1, \tag{4.2}$$

where Y us the Yang–Yang function, given explicitly by

$$\begin{aligned} Y(\lambda) = \frac{L}{2\pi}\sum_{i=1}^{N}&(\lambda_i - i/2)(\log(\lambda_i - i/2) - 1) - (\lambda_i + i/2)(\log(-\lambda_i - i/2) - 1) \\ &- \frac{1}{2\pi}\sum_{i,j=1}^{N}(\lambda_i - \lambda_j + i)(\log(\lambda_i - \lambda_j + i) - 1) + \sum_{j=1}^{N}\lambda_j\left(n_j - \frac{\vartheta}{2\pi}\right). \end{aligned} \tag{4.3}$$

This result we now want to compare with the $\mathcal{N} = (2,2)$ gauge theory with gauge group $U(\widetilde{N})$, one adjoint mass $\widetilde{m}^{\mathbf{adj}}$ and $\widetilde{L}$ fundamental and anti-fundamental fields Q_i, $\overline{Q}_i$ with twisted masses $\widetilde{m}^{\mathbf{f}}$, $\widetilde{m}^{\bar{\mathbf{f}}}$. The effective twisted superpotential for this theory is

$$\begin{aligned} \widetilde{W}_{\mathbf{eff}} = \frac{1}{2\pi}\sum_{i=1}^{\widetilde{N}}\sum_{k=1}^{\widetilde{L}}&(\sigma_i + m_k^{\mathbf{f}})(\log(\sigma_i + m_k^{\mathbf{f}}) - 1) - (\sigma_i + m_k^{\bar{\mathbf{f}}})(\log(-\sigma_i + m_k^{\bar{\mathbf{f}}}) - 1) \\ &- \frac{1}{2\pi}\sum_{i,j=1}^{\widetilde{N}}(\sigma_i - \sigma_j + m^{\mathbf{adj}})(\log(\sigma_i - \sigma_j + m^{\mathbf{adj}}) - 1) - i\tau\sum_{j=1}^{\widetilde{N}}\sigma_j, \end{aligned} \tag{4.4}$$

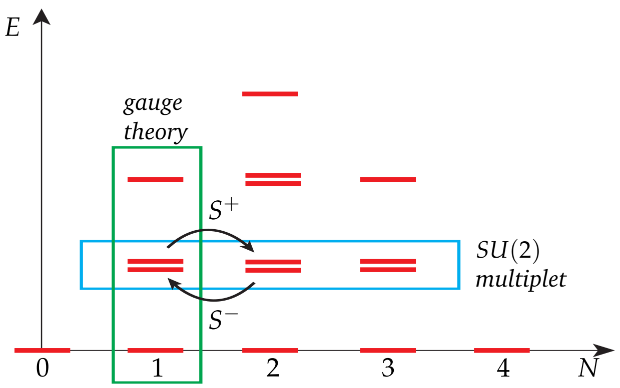

Figure 4.1: The $SU(2)$ symmetry on the $\text{XXX}_{1/2}$ spin chain for $L = 4$. The horizontal arrows show the action of the $S^{\pm}$ operators, changing the magnon number N, preserving the energy. The spectrum can be organized into multiplets of $SU(2)$ (horizontal box) or by magnon number (vertical box).

where the first term comes from the fundamental fields, the second from the anti-fundamental fields, and the third from the adjoint fields. Comparing with Eq. (4.3), we find the two expressions to be the same with the following identifications:

$$\sigma_i = \lambda_i, \qquad \widetilde{N} = N, \tag{4.5}$$

$$m_k^{\mathbf{f}} = -i/2, \qquad m_k^{\bar{\mathbf{f}}} = -i/2 \tag{4.6}$$

$$m^{\mathbf{adj}} = i, \qquad \widetilde{L} = L, \tag{4.7}$$

$$t = \frac{1}{2\pi}\vartheta + in. \tag{4.8}$$

The magnon number N corresponds to the number of colors of the $U(N)$ gauge theory, while the number of flavors corresponds to the length of the spin chain. Since we can identify Y and $\widetilde{W}_{\mathbf{eff}}$, the equations for the spectrum of the N–magnon sector and for the ground states of the $U(N)$ supersymmetric gauge theory are identified as well. The supersymmetric ground states and the N–particle Bethe states are in one-to-one correspondence, this is the main statement of the gauge/Bethe correspondence..

We have seen that in order to obtain the full spectrum of the spin chain, we must solve the Bethe equations for all magnon subsectors, $N = 0, 1, \ldots, L$. Taking the correspondence seriously, we should thus also consider gauge theories with different numbers of colors together, *i.e.* $U(1), \ldots, U(L)$. Let us consider the action of the symmetry group of the integrable model. For concreteness, we take an $\text{XXX}_{1/2}$ spin chain of length $L = 4$, see Figure 4.1. We see that the $S^{\pm}$ operators of $SU(2)$ act horizontally between states with different N, preserving the energy. The full spectrum of the spin chain is thus organized horizontally into $SU(2)$ multiplets. The gauge/Bethe correspondence, on the other hand, identifies states in an N–magnon subsector with the ground states of a gauge theory, slicing the spectrum up vertically. The action of $SU(2)$ is thus a symmetry between gauge theories with different numbers of colors. This can bee seen with an obvious example: for the spin chain, the physics is the same if we use the reference state $|\Omega\rangle = |\uparrow \ldots \uparrow\rangle$ or instead $|\downarrow \ldots \downarrow\rangle$. The sector with N down spins starting with $|\Omega\rangle$ and the one with $L - N$ up spins starting from the reference state $|\downarrow \ldots \downarrow\rangle$ are the same. The spectrum of the spin chain has therefore a manifest $N, L - N$ symmetry. This equivalence is reflected on the gauge theory side as the *Grassmannian duality*. The vacuum manifold of the low energy effective gauge theory corresponding to the $\text{XXX}_{1/2}$ chain is the cotangent bundle of the Grassmannian $T^*\text{Gr}(N, L)$, where the Grassmannian is the collection of all linear subspaces of dimension

gauge theory			integrable model
number of nodes in the quiver	r	r	rank of the symmetry group
gauge group at a–th node	$U(N_a)$	N_a	number of particles of species a
effective twisted superpotential	$\widetilde{W}_{\text{eff}}(\sigma)$	$Y(\lambda)$	Yang–Yang function
equation for the vacua	$e^{2\pi\, \mathrm{d}\, \widetilde{W}_{\text{eff}}} = 1$	$e^{2\pi\imath\, \mathrm{d}\, Y} = 1$	Bethe ansatz equation
flavor group at node a	$U(L_a)$	L_a	effective length for the species a
lowest component of the twisted chiral superfield	$\sigma_i^{(a)}$	$\lambda_i^{(a)}$	rapidity
twisted mass of the fundamental field	$\widetilde{m}_k^{\mathrm{f}(a)}$	$\frac{\imath}{2}\Lambda_k^a + \nu_k^{(a)}$	highest weight of the representation and inhomogeneity
twisted mass of the anti–fundamental field	$\widetilde{m}_k^{\bar{\mathrm{f}}(a)}$	$\frac{\imath}{2}\Lambda_k^a - \nu_k^{(a)}$	highest weight of the representation and inhomogeneity
twisted mass of the adjoint field	$\widetilde{m}^{\mathrm{adj}(a)}$	$\frac{\imath}{2}C^{aa}$	diagonal element of the Cartan matrix
twisted mass of the bifundamental field	$\widetilde{m}^{\mathrm{b}(ab)}$	$\frac{\imath}{2}C^{ab}$	non–diagonal element of the Cartan matrix
FI–term for $U(1)$–factor of gauge group $U(N_a)$	τ_a	$\hat{\vartheta}^a$	boundary twist parameter for particle species a

Table 4.1: Dictionary in the Gauge/Bethe correspondence.

N of a vector space of dimension L:

$$\mathrm{Gr}(N,L) = \{W \subset \mathbb{C}^L | \dim W = N\}, \tag{4.9}$$

$$T^*\mathrm{Gr}(N,L) = \{(X,W), W \in \mathrm{Gr}(N,L),\ X \in \mathrm{End}(\mathbb{C}^L) |\ X(\mathbb{C}^L) \subset W, X(W) = 0\}. \tag{4.10}$$

The Grassmannian duality states that there is an isomorphism between $\mathrm{Gr}(N,L)$ and $\mathrm{Gr}(L-N,L)$, thus linking the ground states of the low energy $U(N)$ and $U(L-N)$ gauge theories.

The integrable structure of the spin chain remains hidden on the gauge theory side of the correspondence as long as the gauge theories with different numbers of colors are considered separately. A mathematical framework that unifies these gauge theories in a meaningful way is Ginzburg's *geometric representation theory*.

We have studied only the simplest example of the correspondence involving the $\mathrm{XXX}_{1/2}$ spin chain, but the scope of the gauge/Bethe correspondence is much larger. In this lecture series, we cannot make recourse to the algebraic Bethe ansatz and the full machinery of group/representation theory to study the more general examples. We will nonetheless have a quick look at the general dictionary between gauge theory and spin chain parameters, see Table 4.1. In general, we are dealing with a quiver gauge theory, which can be summarized by a graph, see Fig. 4.2. The black nodes represent gauge groups, arrows between the nodes correspond to bifundamental fields, arrows from a node to itself indicate adjoint fields, white nodes represent flavor groups and the dashed arrows between flavor and gauge nodes represent fundamental and anti-fundamental fields. Using Table 4.1, the

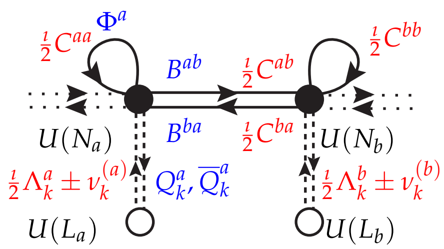

Figure 4.2: Example quiver diagram for the Gauge/Bethe correspondence. Gauge groups are labeled in black, matter fields in blue, the corresponding twisted masses in red.

x	0	1	2	3	4	5	6	7	8	9
D2–brane	×	×	ϕ				×		σ	
NS5–brane	×	×	×	×					×	×
D4–brane	×	×		×					×	×

Table 4.2: Brane set-up for 2d gauge theory

values of the twisted masses that we recovered from the correspondence can be traced back to the Cartan matrix of $SU(2)$,

$$C^{ab} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{4.11}$$

the fact that we had the fundamental representation at every site, so $\Lambda_k = 1$, and the absence of inhomogeneities.

Advanced example: string theory construction for deformed gauge theories. In general, gauge theories can be realized in string theory by placing branes into the ten dimensional string theory background (bulk). A D–brane is an extended object in string theory on which open strings can end with Dirichlet boundary conditions. A Dp–brane has p spatial dimensions and and one time dimension. The fluctuations of a D–brane encode the degrees of freedom of low energy gauge theory.

A two-dimensional gauge theory can be realized in two ways:

- by placing a D1–brane into the bulk.
- by placing a D2–brane between two parallel Neveu–Schwarz (NS) 5–branes.

The difference between the two possibilities is the amount of supersymmetry that is preserved. The ten-dimensional background preserves maximally 32 real supercharges. The D–branes break half of them. The NS5–branes break another half of them. We aim to realize the gauge theory corresponding to the $\mathrm{XXX}_{1/2}$ spin chain. We use the following (preliminary) brane set-up, see Table 4.2, where the x's mark the directions in which the branes are extended. In order to also capture the flavor group present in the gauge/Bethe correspondence, we need to introduce a stack of L D4–branes. The NS5–branes are localized in the directions 4567. Since the D2–brane is extended in a finite interval in x^6, the gauge theory is effectively two-dimensional. The D2 can move along the NS5–brane in the directions x^2, x^3 and x^8, x^9. The fluctuations in these directions correspond to the

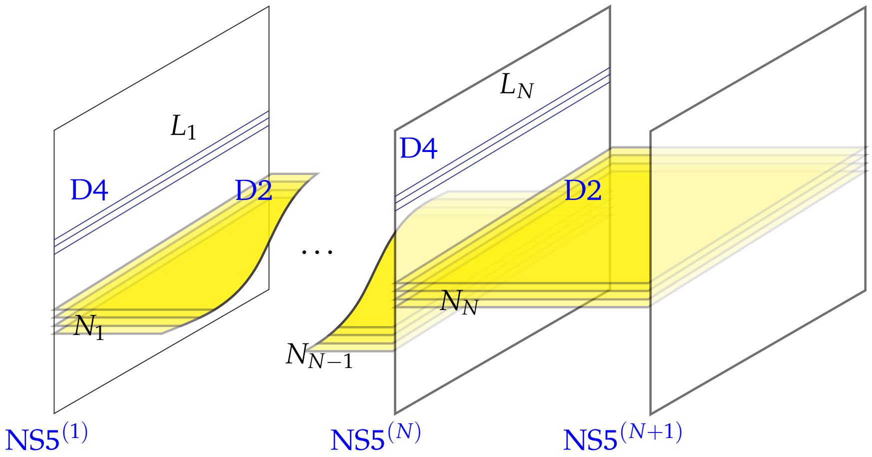

Figure 4.3: Brane cartoon for the $U(N)$ model

x	0	1	2	3	4	5	6	7	8	9
	(ρ_1, θ_1)		(ρ_2, θ_2)		(ρ_3, θ_3)		(ρ_4, θ_4)		v	
fluxbrane	ϵ_1		ϵ_2		ϵ_3		ϵ_4		$\circ$	$\circ$

Table 4.3: Coordinate names and ϵ–deformed directions. Circles denote periodic Melvin directions.

complex scalar fields ϕ, σ of the gauge theory. A single D2–brane leads to a $U(1)$ gauge theory, while N coincident D-branes give rise to the gauge group $U(N)$.

The brane set-up also encodes the symmetry group of the corresponding spin chain. D2–branes suspended between two parallel NS5–branes corresponds to the symmetry group $SU(2)$. For $SU(N)$, we need a set-up with $N+1$ parallel NS5–branes with stacks of D2s between them, see Figure 4.3. Let us study the matter content of the gauge theory. Open strings beginning and ending on the same stack of D2s correspond to adjoint fields. Open strings going from a stack of D2s to the neighboring stack correspond to bifundamental fields, and open strings going from the D2s to the D4s correspond to the fundamental and anti-fundamental fields.

The current string theory construction does not include twisted masses for the gauge fields yet. In order to introduce them, we need to deform the ten-dimensional string theory background.

In oder to describe the background, we divide ten dimensional flat space into four planes each parameterized by a radial coordinate ρ_i and an angular coordinate θ_i, while the x_8, x_9–directions form are periodic, see Table 4.3. Each of the four planes can in principle be deformed via a deformation parameter ϵ_i. To introduce the deformation, we impose the following identifications:

$$\begin{cases} \tilde{x}^8 \simeq \tilde{x}^8 + 2\pi\tilde{R}_8 n_8 \\ \theta_k \simeq \theta_k + 2\pi\epsilon_k^R \tilde{R}_8 n_8 \end{cases} \qquad \begin{cases} \tilde{x}^9 \simeq \tilde{x}^9 + 2\pi\tilde{R}_9 n_9 \\ \theta_k \simeq \theta_k + 2\pi\epsilon_k^I \tilde{R}_9 n_9 \end{cases} \tag{4.12}$$

where $k = 1, 2$, $n_8, n_9 \in \mathbb{Z}$, $\epsilon_k^{R,I} \in \mathbb{R}$ and $\theta_1 = \arctan \tilde{x}^1/\tilde{x}^0$, $\theta_2 = \arctan \tilde{x}^3/\tilde{x}^2$ are independently 2π–periodic variables. The result is the so-called *Melvin* or *fluxbrane* background with complex deformation parameters $\epsilon_k = \epsilon_k^R + \epsilon_k^I$. In order to preserve part of the supersymmetry, the ϵ_k have to fulfill the relation

$$\sum \pm\epsilon_k = 0. \tag{4.13}$$

fluxtrap				ϵ_i	ϵ_j
D–brane	$\times$	$\times$	$\times$	ϕ_i	

Table 4.4: D–brane configuration in the fluxtrap corresponding to a twisted mass ϵ_i for the field ϕ_i.

fluxtrap	ϵ_i		ϵ_j	
D–brane	$\times$	$\times$	$\times$	$\times$

Table 4.5: D–brane configuration in the fluxtrap corresponding to a Ω–deformed gauge theory.

After performing two T–dualities in the periodic directions, we arrive at the *fluxtrap* background. If for simplicity we set $\epsilon_1 \in \mathbb{R}$, $\epsilon_2 \in \operatorname{im}\mathbb{R}$, $\epsilon_3 = \epsilon_4 = 0$, the bulk fields of this background takes the form

$$\mathrm{d}\,s^2 = \mathrm{d}\,\rho_1^2 + \frac{\rho_1^2\,\mathrm{d}\,\phi_1^2 + \mathrm{d}\,x_8^2}{1+\epsilon_1^2\rho_1^2} + \mathrm{d}\,\rho_2^2 + \frac{\rho_2^2\,\mathrm{d}\,\phi_2^2 + \mathrm{d}\,x_9^2}{1+\epsilon_2^2\rho_2^2} + \sum_{k=4}^{7}(\mathrm{d}\,x^k)^2, \tag{4.14a}$$

$$B = \epsilon_1 \frac{\rho_1^2}{1+\epsilon_1^2\rho_1^2}\,\mathrm{d}\,\phi_1 \wedge \mathrm{d}\,x_8 + \epsilon_2 \frac{\rho_2^2}{1+\epsilon_2^2\rho_2^2}\,\mathrm{d}\,\phi_2 \wedge \mathrm{d}\,x_9\,, \tag{4.14b}$$

$$\mathrm{e}^{-\Phi} = \sqrt{\left(1+\epsilon_1^2\rho_1^2\right)\left(1+\epsilon_2^2\rho_2^2\right)}\,, \tag{4.14c}$$

where $\phi_k = \theta_k - \epsilon_k^R \tilde{x}^8 - \epsilon_k^I \tilde{x}^9$ are the new 2π–periodic angular coordinates. We see that the following things have happened:

- the metric is no longer flat.
- a B–field has appeared.
- the dilaton is no longer constant.

The type of gauge theory deformation resulting from the fluxtrap background depends on how the D–branes are placed into the fluxtrap with respect to the deformations in the bulk. There are basically two possibilities, which can be combined.

- The background deformations being *orthogonal* to the brane world-volume gives rise to *mass-type* deformations for the scalar fields encoding brane fluctuations in the deformed directions[1], see Table 4.4.
- When the background deformation happens *on* the brane world-volume, the effective gauge theory receives an Ω*–type* deformation where Lorentz invariance is broken, see Table 4.5.

The brane set-up for our 2d gauge theory in Table 4.2 is now supplemented by the background deformation as shown in Table 4.6. The fluxtrap background pins the D2–brane to the origin $x^2 = x^3 = x^4 = x^5 = x^7 = 0$, thus the name "trap". When expanding the Dirac–Born–Infeld action of the D2–brane, it turns out that the deformation parameter $\epsilon_1 = -\epsilon_2 = m$ acts as a twisted mass parameter for the resulting gauge theory. Thus the above brane set-up in the fluxtrap background realizes the $\mathcal{N} = (2,2)$ gauge theory appearing in the gauge/Bethe correspondence.

[1] Deformed directions away from the brane world-volume without an associated scalar field result in R–symmetries for the gauge theory.

x	0	1	2	3	4	5	6	7	8	9
fluxtrap			ϵ_1		ϵ_2				$\circ$	$\circ$
D2–brane	$\times$	$\times$	ϕ				$\times$		σ	
NS5–brane	$\times$	$\times$	$\times$	$\times$					$\times$	$\times$

Table 4.6: D2–brane set-up and its scalar fields in the fluxtrap background, $\epsilon_1 = -\epsilon_2 = m$

Literature. The 2d gauge/Bethe correspondence was introduced in [1, 2]. The dictionary is explained in detail in [3, 4] and the string theory realization is summarized in the review paper [5].

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