Exercise 1. *Bloch Oscillations*

In the quasi-classical description of a wave-packet peaked around some quasi-momentum $\hbar k$ the group velocity is given by

$$\dot{r} = \frac{1}{\hbar}\frac{\partial \varepsilon_k}{\partial k}, \tag{1}$$

while the change of the quasi-momentum is given by

$$\hbar \dot{k} = F_{\text{ext}}, \tag{2}$$

with F_{ext} the force due to applied external fields (in addition to the periodic potential).

(a) We focus on the one-dimensional tight-binding model with dispersion relation

$$\varepsilon_k = -2\mathrm{t}\cos(ka). \tag{3}$$

Show that a uniform electric field does not accelerate the electrons but lets them oscillate around some fixed position. This means that, for sufficiently large fields, all metals would behave like insulators. Why has this effect never been seen in normal metals? What would change if we considered semiconductor superlattices instead of metals?

Hint: Knowing that metals have a relaxation time of the order of $\sim 10^{-14}$s and a lattice constant $a \approx 1$Å, estimate the minimum field needed for observing Bloch oscillations. Bear in mind that the quasi-classical approximation breaks down in the case of strong fields. In order to know why/how Bloch oscillations were detected in semiconductor superlattices you might want to read the paper by C. Waschke *et al.*, Phys. Rev. Lett. **70**, 3319 (1993) and Physics Today **46**(6), 34 (1993).

(b) We now add a small damping term to Eq. (2) and analyze the consequences. The rate of change of the quasi-momentum is thus given by

$$\hbar \dot{k} = F_{\text{ext}} - \frac{m\dot{r}}{\tau}, \tag{4}$$

where τ is the relaxation time. Show that this damping can lead to a vanishing of the oscillations and thus to a stationary solution. What is the corresponding condition and how does the stationary solution look like? Calculate then analytically $k(t)$ for both situations to verify your considerations.

Exercise 2. *One-Dimensional Model of a Semiconductor*

Let us consider electrons moving on a one-dimensional chain. We use the so-called tight-binding approximation. Thus, we assume that each atom has a localized electron state and that the electrons are able to hop between neighboring atoms. This hopping process describes the kinetic energy term.

It is most convenient to use a second-quantized language. For simplicity, we assume the electrons to be spinless fermions. Let c_i and $c_i^\dagger$ be the creation and annihilation operators for an electron at site i, respectively. The overlap integral between neighboring electron states is denoted by t. Then, the kinetic energy operator is written as

$$H_0 = -\mathrm{t}\sum_i \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right). \tag{5}$$

We assume that the chain contains N atoms and in the following we set the lattice constant $a = 1$. As a second step, we consider an alternating bipartite lattice which we model by a potential of the form

$$V = v \sum_i (-1)^i \; c_i^\dagger c_i. \tag{6}$$

(a) Consider first the case $v = 0$. Show that the states created by

$$c_k^\dagger = \frac{1}{\sqrt{N}} \sum_j e^{-ikj} c_j^\dagger \tag{7}$$

are eigenstates of H_0 with energy $\epsilon_k = -2\mathrm{t} \cos k$. Here, k belongs to the first Brillouin zone $[-\pi, \pi)$.

(b) For $v \neq 0$ the creation operators for the new eigenstates can be obtained by means of a so-called *Bogoliubov transformation* which we write as

$$a_k^\dagger = u_k c_k^\dagger + v_k c_{k+\pi}^\dagger, \qquad\qquad b_k^\dagger = v_k c_k^\dagger - u_k c_{k+\pi}^\dagger \tag{8}$$

where $u_k^2 + v_k^2 = 1$ (both u_k and v_k may be assumed to be real) for all k in the reduced Brillouin zone $[-\pi/2, \pi/2)$. Diagonalize the Hamiltonian and show that it can be written in the form

$$H_0 + V = \sum_{k \in [-\frac{\pi}{2}, \frac{\pi}{2})} \left(-E_k a_k^\dagger a_k + E_k b_k^\dagger b_k \right), \qquad E_k = \sqrt{\epsilon_k^2 + v^2}. \tag{9}$$

(c) Consider now the ground state of the half-filled chain ($N/2$ electrons). What is the difference between the cases (a) and (b)?